A CONVENTIONAL POST-NEWTONIAN MACH EFFECT

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Realizing that Jim has derived a post-Newtonian field equation (equation 5 of Estes Park Quick Study VI), and that standard GR provides a prescription for generating post-Newtonian field equations, I wanted to do a straightforward calculation in GR to see what the equation corresponding to (VI-5) would be. This approach does not require Jim's ansatzes 1, 2, or 3. It departs merely from textbook results in GR.

So let us start with the field equations. In the linear theory of GR, the metric $g_{\mu\nu}$ is decomposed into a Minkowski piece $\eta_{\mu\nu}$ and a small perturbation: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu} \ll 1$.

Standard texts give the field equations for the linear theory in a convenient gauge. Weinberg's expression (10.1.10) in harmonic gauge is:

$$\frac{\partial^2 h_{\mu\nu}}{c^2 \partial t^2} - \nabla^2 h_{\mu\nu} = \frac{16\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^\alpha_\alpha\right) \tag{1}$$

The Newtonian limit is recovered from the linear theory by considering the time-time (0-0) component, which relates h_{00} to $T_{00} \sim \rho c^2$. This would seem to be the reasonable departure point to connect to (VI-5). The difficulty is to recover the peculiar mass time derivatives in (VI-5). We see that the standard linear equation (1) has no time derivatives of mass, but instead time derivatives of the field.

The interaction between matter and fields is not solely described by the field equations, which describe how fields arise from matter. We also need to look at the equations of motion, which describe how fields influence matter. The equation of motion for GR is called the geodesic equation, and it describes how matter is influenced by gravity. It describes the precession of the perihelion of Mercury, for example. It is in terms of the 4-velocity $U^\mu \equiv dx^\mu/d\tau = p^\mu/m$:

$$U^\alpha \nabla_\alpha p^\mu = U^\alpha\left(\frac{\partial p^\mu}{\partial x^\alpha} + \Gamma^\mu_{\alpha\beta}p^\beta\right) = \frac{dp^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}U^\alpha p^\beta = 0 \tag{2}$$

Equation (2), along with the Einstein field equations, is the whole content of GR. The non-relativistic, post-Newtonian limits of (2) are given by Weinberg in section 3.4, and by Schutz in section 7.2. In this case, the spatial components of the 4-momentum p^μ are assumed much less than the time component. Then the non-relativistic limit of the geodesic equation (2) is:

$$\frac{dp^\mu}{d\tau} + \Gamma^\mu_{00}U^0 p^0 = Order(v/c) \tag{3}$$

The linear affine connection is given by

$$\Gamma^\mu_{00} \simeq \frac{1}{2}\eta^{\mu\nu}\left(2\frac{\partial h_{0\nu}}{c\partial t} - \frac{\partial h_{00}}{\partial x^\nu}\right) \tag{4}$$

Returning to the field equations, consider a simple stress-energy tensor, for a cold fluid or dust: $T_{\mu\nu} = \rho W_\mu W_\nu$, where $\int \rho W^\alpha dV \equiv \Sigma p^\alpha$ of the particles. Then

$$T^\alpha_\alpha \simeq \rho\left(\frac{cdt}{d\tau}\right)^2 = \rho W_0^2 \tag{5}$$

where again we ignore the spatial components of the 4-momentum relative to the time component, this time now for the bulk matter.

Let us put this together, setting $h_{00} \equiv -2\phi/c^2$, to accord with the usual identification of the Newtonian potential ϕ with the time-time component of the metric perturbation. Then the linear field equation (1) can be written:

$$\nabla^2\phi - \frac{\partial^2\phi}{c^2\partial t^2} = \frac{4\pi G}{c^2}\rho W_0^2 \tag{6}$$

Likewise, the linear equation of motion (3) for the time component p^0 of the 4-momentum is (cf. also Carroll 7.23):

$$\frac{U^0}{c}\frac{\partial p^0}{\partial t} = \frac{U^0}{c^3}\frac{\partial \phi}{\partial t}p^0 \tag{7}$$

Equation (7) is typically dropped in textbook treatments of linearized gravity, on the assumption that the Newtonian limit should be time-independent. Those treatments are typically interested in the spatial pieces of (3) that show trajectories under gravitational influence. Allowing this time dependence is a nice insight that Jim has brought to this discussion.

Equation (7) allows us to write the time derivative of the field in terms of particle energy:

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{c^2}{p^0}\frac{\partial p^0}{\partial t}\right) = \frac{c^2}{p^0}\frac{\partial^2 p^0}{\partial t^2} - \left(\frac{c}{p^0}\right)^2\left(\frac{\partial p^0}{\partial t}\right)^2 \tag{8}$$

so that the field equation (6) now becomes:

$$\nabla^2 \phi = \frac{4\pi G}{c^2}\rho W_0^2 + \frac{1}{p^0}\frac{\partial^2 p^0}{\partial t^2} - \left(\frac{1}{p^0}\right)^2\left(\frac{\partial p^0}{\partial t}\right)^2 \tag{9}$$

This is strikingly similar to (VI-5). To make the identification complete, put $p^0 = mc$ and $W^0 = c$, and convert the mass factors to mass density:

$$\nabla^2 \phi = \frac{4\pi G}{c^2}\rho W_0^2 + \frac{1}{\rho}\frac{\partial^2 \rho}{\partial t^2} - \left(\frac{1}{\rho}\right)^2\left(\frac{\partial \rho}{\partial t}\right)^2 \tag{10}$$

Furthermore, this recovers the 2nd and 3rd terms on the RHS of (VI-5) if Jim's original substitution is used in those terms to set ratios of $\phi/c^2 \to 1$. Missing from (10) relative to Jim's (VI-5) are the quadratic time derivatives of the field. Neither of those terms are important to Jim's Mach effect, so it appears they are dispensible parts of his theory. I like this simple derivation here because it reproduces the essential parts of (VI-5) without any assumptions or without the uncertainty of where to substitute for c^2.

Indeed, in hindsight we can recover equation (10) from Jim's equation (VI-4) if the 3rd ansatz is dropped. Taking straightforward derivatives of the first term on the LHS of (VI-4) yields exactly equation (10). This seems to imply that the 3rd ansatz is outside GR, and it accounts for the extra time derivative terms in Jim's derivation.

Jim feels the extra time derivatives are important, especially the piece of the d'Alembertian, to make the Mach equation look relativisitic. I am not so bothered by this because we know the Newtonian limit of GR is not covariant, and we know the linear limits of GR are not covariant. We can still get relativistic effects to a certain order, but without a covariant equation.