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A new theory of gravitation

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A new theory of gravitation is developed. The theory is equivalent to that of Einstein in the description of macroscopic phenomena, and hence the situation is the same so far as the classical tests of general relativity are concerned. The new theory differs in its global implications, however. There are two main differences of principle. In the usual theory, the negative sign of the constant of proportionality $-8\pi G$ which appears in the field equations $R^{ik}-\frac{1}{2}g^{ik}R = -8\pi GT^{ik}$ is chosen arbitrarily. In the present theory there is no such ambiguity; the sign must be minus. Further, the magnitude of G follows from a determination of the mean density of matter, thereby enabling the cosmologist to know how hard he will hit the ground if he is unfortunate enough to fall over a cliff. The second point of principle is that the equation $R_{ik} = 0$ for an empty world in Einstein theory becomes meaningless; there is no such thing as an 'empty' world; in the present theory emptiness demands no world at all. Nor can there be a world containing a single particle, the least number of particles is two.

INTRODUCTION

The concept of 'action at a distance' was first introduced into physics by Newton. In Newtonian theory the gravitational interaction between two particles propagates instantaneously. The success of the gravitational theory prompted attempts to formulate an 'action at a distance' theory of electromagnetism. In 1845 Gauss went so far as to describe an action propagating with a finite speed 'as in the case with light'. His work was not completed, however, and no further substantial progress was made until Maxwell formulated the field theory of electromagnetism in 1875. Since then, field theory has gained wide support in physics and has withstood, with reasonable success, the two revolutions brought about by the theory of relativity and by the quantum theory.

The success of field theory has overshadowed the 'action at a distance' theories, although, ironically, we nowadays need have no difficulty with the problem that seemed so worrying to Newton and his followers, namely the mystery of how particles manage to act on each other when they are at a distance apart. We now know that particle couplings are propagated essentially along null geodesics—i.e. at *no* distance in the four-dimensional sense. Strictly, the phrase 'action at a distance' should be changed to 'action at no distance'.

Attempts to formulate electromagnetic theory in terms of a direct particle interaction were made by Schwarzschild (1903), Tetrode (1922), and Fokker (1929*a*, *b*, 1932). These attempts gave a consistent mathematical formulation of the ideas expressed by Gauss and gave results in agreement with Maxwell's equations for static and steady-state electromagnetism. Difficulties were encountered in the general description of time-dependent electrodynamics, however. Because of the complete time-symmetry of these theories, the field of an accelerated electron is described by the sum of *half* the usual retarded solution of Maxwell's equations and *half* the advanced solution. The presence of the advanced solution is contrary to experience. Moreover, an accelerated charge in an otherwise empty space experiences no electromagnetic force, whereas a damping force is actually observed.

It was first shown by Wheeler & Feynman (1945, 1949) that both these difficulties can be overcome. They pointed out that the particles we actually observe to radiate are not in an otherwise empty world, so the theoretical result that such particles should not radiate is not necessarily in contradiction with experience. And by a remarkable argument they deduced that a static homogeneous universe of charged particles could produce a reaction equal to half the usual retarded solution due to the accelerated charge *minus* half the advanced solution. This solved both difficulties, since it not only gave the correct reaction force on the particle but the sum of the field due to the particle with the reaction field due to the universe gave the fully retarded field of normal experience.

However, because of the complete time symmetry of the theory it was also possible to obtain a consistent solution corresponding to a fully *advanced* field. To get a definitive result in a static universe, Wheeler & Feynman had to fall back on considerations of statistical mechanics. These considerations were shown to be unnecessary by Hogarth (1962) who found that a definite result could be obtained if the universe were taken as expanding, rather than static. Hoyle & Narlikar (1964*a*) have re-examined this issue and have shown that a self-consistent retarded solution is possible in steady-state cosmology and that a self-consistent advanced solution is possible in the Einstein–de Sitter cosmology.

We are now in a position to explain the motivation of the present paper. The starting-point of our work has been the conviction that it is deeply unsatisfactory to be obliged to make an empirical choice of the retarded solutions of Maxwell's equations, as is necessary in field theory. The situation would still be bad, even if no alternative were available. With the realization that such an empirical choice could be avoided, that indeed full time-symmetry in the solutions could be accepted in an appropriate cosmology, we became convinced that the further consequences of interparticle action should be explored. Evidently, it is not reasonable to dispense with the independent degrees of freedom of the electromagnetic field and yet to retain such degrees of freedom for other fields: if we reduce the electromagnetic field to a direct particle field, we must do the same for other fields. This line of argument was followed in a previous paper (1964*b*), in which we showed that the *C*-field could be represented as a direct particle field. Perhaps the most awkward problem was to determine the gravitational influence of a direct particle field. We found it possible (1964*c*) to work out the effect of the electromagnetic field, obtaining the usual results for the energy-momentum tensor, without it being necessary to assume independent degrees of freedom for the field. These results, taken together, seem to us to provide further incentive to extend the direct action theory. However, the next step involves a radically new departure.

The form of the action so far developed is

$$J = \frac{1}{16\pi G}\int R\sqrt{(-g)}\,\mathrm{d}^4x - \sum_a m_a \int \mathrm{d}a - \sum_{a<b}\sum 4\pi e_a e_b \iint \bar{G}_{i_A i_B}\,\mathrm{d}a^{i_A}\,\mathrm{d}b^{i_B}$$
$$+ f^{-1}\sum_{a<b}\sum \iint \bar{G}_{;\,i_A i_B}\,\mathrm{d}a^{i_A}\,\mathrm{d}b^{i_B}. \qquad (1)$$

The third and fourth terms are analogous to each other, but the first and second terms bear no affinity to them. Indeed, only long familiarity with the first and second terms reconciles us to their conjunction, the first a quadruple integral; the second, a single line integral term. From the present point of view, both are objectionable, the first being similar to the action of a field with independent degrees of freedom, the second being a self-action term.

The second term is derived from Galileo's concept of inertia and has been present in physics, in one form or another, since before Newton. Because Einstein retained this traditional term, he was led to add the first term, thereby creating an implausible combination. Our point of view is that neither term is correct, that only double integral expressions of the type of the third and fourth terms should be included in the action. In our first attempt to follow this argument we tried replacing each of the first and second terms by double integral expressions, and it was only after failure to find satisfactory forms that we realized that only *one* double integral expression is needed, that the inelegance of the first and second terms arises from an artificial separation of a single term into two parts. In what follows we shall therefore construct the purely gravitational theory (the equivalent of the first and second terms of (1)) from a single expression. With this done, with the purely gravitational theory given by one double integral expression, the obvious next step is to attempt a further simplification of the action, by a further collapsing of terms into each other: we shall not tackle this problem in the present paper, however. We content ourselves with the remark that we believe this to be the correct path towards a united theory of gravitation and electricity.

Mass as a direct particle field

In order to convert the single line integral $\int m_a\,\mathrm{d}a$ into a sum of double line integrals we are guided by the following considerations:

(1) The mass m_a must become a direct particle field, it must arise from all other particles in the universe.

(2) Since mass is scalar we expect it to arise through a scalar Green function.

(3) The action must be symmetric between any pair of particles.

We write

$$\int m_a\,\mathrm{d}a = -\lambda \sum_{b\neq a} \iint \tilde{G}(A,B)\,\mathrm{d}a\,\mathrm{d}b, \qquad (2)$$

where λ is a coupling constant and $\tilde{G}(A,B)$ is a scalar Green function as yet unspecified. It satisfies $\tilde{G}(A,B) = \tilde{G}(B,A)$. The mass function at a general point x due to a is defined by

$$m^{(a)}(x) = -\lambda \int \tilde{G}(X,A)\,\mathrm{d}a. \qquad (3)$$

The mass function therefore varies from point to point.

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Omitting the electromagnetic field and the C-field, which we shall do throughout most of this paper, the action is

$$J = -\sum_a \frac{1}{2}\int m_a\,\mathrm{d}a = \lambda \sum_{a<b}\sum \iint \tilde{G}(A,B)\,\mathrm{d}a\,\mathrm{d}b. \tag{4}$$

The factor $\frac{1}{2}$ comes in because each $\tilde{G}(A,B)$ is shared by two particles a, b. This makes no difference to the equations of motion of particles as will be seen later.

The purely gravitational theory is to be obtained from this very simple form, once the wave equation for $\tilde{G}$ has been specified. The most general wave equation is

$$g^{i_X k_X}\tilde{G}(X,A)_{;i_X k_X} + \mu R\tilde{G}(X,A) = -(-\bar{g})^{-\frac{1}{2}}\,\delta^{(4)}_{(X,A)}, \tag{5}$$

in which μ is a constant. Analogy with the electromagnetic field suggests that μ should be taken as $\frac{1}{6}$, since the wave equation is then conformally invariant (Penrose 1963). At a later stage we shall give a further reason for the choice $\mu = \frac{1}{6}$. This reason will become more evident if for the moment we work with μ unspecified.

Using Riemannian co-ordinates at A, and arranging that particle a is at rest when at A, we expect $\int \tilde{G}\,\mathrm{d}a \approx -1/r$ at any x in the same time section as A, r being the three-dimensional distance of x from A. Hence we require $\lambda > 0$ in order that $m^{(a)}$ be positive. This expectation will be confirmed at a later stage.

The next step is to vary the geometry in a finite volume V, $g_{ik} \to g_{ik} + \delta g_{ik}$ with $\delta g_{ik} = 0$ on the boundary of V. By using the method developed in a previous paper (1964c) it is possible, as will be shown in the next section, to obtain

$$\delta J = 2\lambda \int P^{ik}\,\delta g_{ik}\sqrt{(-g)}\,\mathrm{d}^4y, \tag{6}$$

in which P^{ik} is a symmetrical tensor. So far, we have been concerned only with making definitions and with symbol manipulation. The formalism becomes a physical theory as soon as we assert that $\delta J = 0$ for all such variations of geometry. This requires

$$P^{ik} = 0, \tag{7}$$

and these are the field equations of the new theory.

It is possible to deduce the equations of motion of a particle from (7). Alternatively, the equations of motion of particle a can be obtained by varying the world-line of a, but keeping the geometry fixed, and by again requiring $\delta J = 0$. The two procedures are equivalent because a variation of the world-line of a,

$$x^i(a) \to x^i(a) + \delta x^i(a)$$

with g_{ik} fixed, can also be represented by $x^i(a) \to x^i(a)$, $g_{ik} \to g_{ik} + \delta g_{ik}$ with appropriate δg_{ik}. To obtain the equations of motion it is easier to consider

$$x^i(a) \to x^i(a) + \delta x^i(a),$$

g_{ik} fixed. It is worth noting that the change in the action (4) can then be written as $-\delta\int m_a\,\mathrm{d}a$. The factor $\frac{1}{2}$ thus disappears. A calculation along well-known lines leads to

$$\frac{\mathrm{d}}{\mathrm{d}a}\left(m_a\frac{\mathrm{d}a^i}{\mathrm{d}a}\right) + m_a\Gamma^i_{kl}\frac{\mathrm{d}a^k}{\mathrm{d}a}\frac{\mathrm{d}a^l}{\mathrm{d}a} - g^{ik}\frac{\partial m_a}{\partial a^k} = e_a\sum_{b\neq a} F^{(b)i}{}_k\frac{\mathrm{d}a^k}{\mathrm{d}a}, \tag{8}$$

in which $m_a = \sum_{b\neq a} m^{(b)}(a^i)$ and in which we have included the Lorentz force for the sake of completeness. Although we are not considering the C-field in the present paper, it may be noted that the condition to be satisfied at an end of a broken world-line is unchanged from that given previously (1964*b*), viz.

$$m_a \frac{\mathrm{d}a^i}{\mathrm{d}a} + e_a \sum_{b \neq a} A^{(b)i} - \sum_{b \neq a} C^{(b)i} = 0. \tag{9}$$

The world-lines of particles are not in general geodesics in the new theory, even in the absence of an electromagnetic field. At first sight this might seem an overriding objection to the present development. It is important, however, not to prejudge the issue. The mass m_a associated with particle a arises from all other particles in the universe. It is possible to contemplate a cosmological situation in which the latter are so distributed that m_a turns out independent of position, in which case the geodesic equation is recovered. We see therefore that cosmological issues arise at an early stage in the present theory, as indeed must be the case if 'mass' is not a form of self-action. We shall resolve this question at a later stage.

The loss of the geodesic property might appear as a loss of elegance. To this criticism we reply that there may be no loss in the long run. In our view, a worthwhile unification of gravitation and electromagnetism was always impossible so long as the geodesic property applied of necessity in the purely gravitational theory. It seems to us that a stumbling block may have been removed, and that the situation may really prove to be the other way round.

The gravitational equations

Following the method developed previously (1964*c*) we first consider the change in $\tilde{G}(A,B)$ due to an infinitesimal change δg^{ik} in g^{ik} over a finite volume V, with $\delta g^{ik} = 0$ on the boundary of V.

The equation satisfied by $\tilde{G}(X,A)$ can be written in the form

$$\frac{\partial}{\partial x^{iX}}\left[\sqrt{(-g)}\, g^{iXkX} \frac{\partial \tilde{G}(X,A)}{\partial x^{kX}}\right] + \mu R \sqrt{(-g)}\, \tilde{G}(X,A) = -\delta^{(4)}_{(X,A)}. \tag{10}$$

The variation of (10) is

$$\frac{\partial}{\partial x^i}\left[\sqrt{(-g)}\, g^{ik} \frac{\partial \delta\tilde{G}}{\partial x^k}\right] + \mu R \sqrt{(-g)}\, \delta\tilde{G} = -\frac{\partial}{\partial x^i}\left[\delta(\sqrt{(-g)}\, g^{ik}) \frac{\partial \tilde{G}}{\partial x^k}\right] - \mu\delta(R\sqrt{(-g)})\, \tilde{G}, \tag{11}$$

in which the subscript X in the indices has been omitted. Using the original Green function $\tilde{G}$, we get

$$\begin{aligned}\delta\tilde{G}(A,B) &= \int_V \frac{\partial}{\partial x^i}\left[\delta(\sqrt{(-g)}\, g^{ik}) \frac{\partial \tilde{G}(A,X)}{\partial x^k}\right] \tilde{G}(B,X)\, \mathrm{d}^4x \\ &\qquad + \mu \int \delta(R\sqrt{(-g)})\, \tilde{G}(A,X)\, \tilde{G}(B,X)\, \mathrm{d}^4x \\ &= -\int_V \delta(\sqrt{(-g)}\, g^{ik}) \frac{\partial \tilde{G}(A,X)}{\partial x^k} \frac{\partial \tilde{G}(B,X)}{\partial x^i}\, \mathrm{d}^4x \\ &\qquad + \mu \int \delta(R\sqrt{(-g)})\, \tilde{G}(A,X)\, \tilde{G}(B,X)\, \mathrm{d}^4x. \end{aligned} \tag{12}$$

We therefore get

$$\delta J = \lambda\delta \sum_{a<b}\sum \iint \tilde{G}(A,B)\,\mathrm{d}a\,\mathrm{d}b = -\sum_a \int m_a\,\delta(\mathrm{d}a)$$
$$-\frac{1}{\lambda}\sum_{a<b}\sum \int_V \delta(\surd(-g)\,g^{ik})\,m^{(a)}(x)_{;k}\,m^{(b)}(x)_{;i}\,\mathrm{d}^4x$$
$$+\frac{\mu}{\lambda}\sum_{a<b}\sum \int_V \delta(R\surd(-g))\,m^{(a)}(x)\,m^{(b)}(x)\,\mathrm{d}^4x. \tag{13}$$

The first term leads to the familiar energy momentum tensor for a set of particles, $\underset{m}{T^{ik}}$ say. Thus

$$-\sum_a \int m_a\,\delta(\mathrm{d}a) = -\frac{1}{2}\int_V \underset{m}{T^{ik}}\,\delta g_{ik}\surd(-g)\,\mathrm{d}^4x, \tag{14}$$

$$\underset{m}{T^{ik}}(x) = \sum_a \int \delta^{(4)}_{(X,A)}\,[-\bar{g}(X,A)]^{-\frac{1}{2}}\,m_a\,\frac{\mathrm{d}a^{i_A}}{\mathrm{d}a}\,\frac{\mathrm{d}a^{k_A}}{\mathrm{d}a}\,\bar{g}^i_{i_A}\,\bar{g}^k_{k_A}\,\mathrm{d}a. \tag{15}$$

The second term reduces to

$$-\frac{1}{\lambda}\sum_{a<b}\sum \int \delta(\surd(-g)\,g^{ik})\,m^{(a)}_{;k}m^{(b)}_{;i}\,\mathrm{d}^4x$$
$$= -\frac{1}{2\lambda}\sum_{a<b}\sum \int [m^{(a)}_{;i}\,m^{(b)}_{;k} + m^{(a)}_{;k}m^{(b)}_{;i} - g_{ik}\,m^{(a);l}\,m^{(b)}_{;l}]\,\surd(-g)\,\delta g^{ik}\,\mathrm{d}^4x. \tag{16}$$

We now have a rather long computation for the last term of (13),

$$\frac{\mu}{\lambda}\sum_{a<b}\sum \int_V m^{(a)}(x)\,m^{(b)}(x)\,(R_{ik}-\tfrac{1}{2}g_{ik}R)\,\delta g^{ik}\surd(-g)\,\mathrm{d}^4x$$
$$+\frac{\mu}{\lambda}\sum_{a<b}\sum \int_V m^{(a)}(x)\,m^{(b)}(x)\,\delta R_{ik}\,g^{ik}\surd(-g)\,\mathrm{d}^4x, \tag{17}$$

in which the second term requires some reduction. It is easily verified that

$$g^{ik}\delta R_{ik} = -\frac{1}{\surd(-g)}\frac{\partial}{\partial x^l}[\surd(-g)\,w^l], \tag{18}$$

where
$$w^l = g^{ik}\delta\Gamma^l_{ik} - g^{il}\delta\Gamma^k_{ik}. \tag{19}$$

Using the identities

$$\Gamma^k_{ik} = \frac{1}{\surd(-g)}\frac{\partial}{\partial x^i}(\surd(-g)), \quad g^{ik}\,\Gamma^l_{ik} = -\frac{1}{\surd(-g)}\frac{\partial}{\partial x^k}(\surd(-g)\,g^{lk}), \tag{20}$$

we can express w^l in the form

$$w^l = -\delta g^{ik}\,[\Gamma^l_{ik} + \tfrac{1}{2}g_{ik}\,g^{pl}\,\Gamma^q_{pq} - \tfrac{1}{2}g_{ik}\,g^{pq}\,\Gamma^l_{pq}]$$
$$-\frac{1}{\surd(-g)}\frac{\partial}{\partial x^i}[\delta(g^{il}\surd(-g))] - \frac{1}{\surd(-g)}\,g^{il}\frac{\partial}{\partial x^i}(\delta\surd(-g)). \tag{21}$$

For convenience write

$$M = m^{(a)}(x)\,m^{(b)}(x), \quad M_l = \partial M/\partial x^l. \tag{22}$$

Then, on integrating by parts,

$$
\begin{aligned}
\frac{\mu}{\lambda}\int_V \delta R_{ik}\, g^{ik}\sqrt{(-g)}\, m^{(a)}(x)\, m^{(b)}(x)\,\mathrm{d}^4x
&= \frac{\mu}{\lambda}\int_V w^l M_l \sqrt{(-g)}\,\mathrm{d}^4x \\
&= -\frac{\mu}{\lambda}\int_V \delta g^{ik}[\Gamma^l_{ik} + \tfrac{1}{2} g_{ik}\, g^{pl}\,\Gamma^q_{pq} - \tfrac{1}{2} g_{ik}\, g^{pq}\,\Gamma^l_{pq}]\, M_l \sqrt{(-g)}\,\mathrm{d}^4x \\
&\qquad + \frac{\mu}{\lambda}\int_V \left[\delta(g^{il}\sqrt{(-g)})\,\frac{\partial M_l}{\partial x^i} - \delta(\sqrt{(-g)})\,\frac{\partial}{\partial x^i}(g^{il} M_l)\right]\mathrm{d}^4x \\
&= -\frac{\mu}{\lambda}\int_V \delta g^{ik}\Big[\{\Gamma^l_{ik} + \tfrac{1}{2} g_{ik}\, g^{pl}\,\Gamma^q_{pq} - \tfrac{1}{2} g_{ik}\, g^{pq}\,\Gamma^l_{pq}\}\, M_l \\
&\qquad - \left\{\frac{\partial M_i}{\partial x^k} - \tfrac{1}{2} g^{pl} g_{ik}\,\frac{\partial M_l}{\partial x^p} - \tfrac{1}{2} g_{ik}\,\frac{\partial}{\partial x^p}(g^{pl} M_l)\right\}\Big]\sqrt{(-g)}\,\mathrm{d}^4x. \qquad (23)
\end{aligned}
$$

It is of interest that this apparently complex expression collapses into the following compact form,

$$-\frac{\mu}{\lambda}\int_V \delta g^{ik}[g_{ik}\, g^{pq} M_{;pq} - M_{;ik}]\sqrt{(-g)}\,\mathrm{d}^4x, \qquad (24)$$

where

$$M_{;ik} = m^{(a)}_{;ik}\, m^{(b)} + m^{(a)}\, m^{(b)}_{;ik} + m^{(a)}_{;i}\, m^{(b)}_{;k} + m^{(a)}_{;k}\, m^{(b)}_{;i}. \qquad (25)$$

We are now in a position to return to the simplified form taken by (13). Combining (14), (16), (17) and (24), we have

$$
\begin{aligned}
\delta J = &-\frac{1}{2}\int_V \delta g_{ik}\, \underset{m}{T}^{ik}\sqrt{(-g)}\,\mathrm{d}^4x \\
&-\frac{1}{2\lambda}\sum_{a<b}\sum\int_V \delta g^{ik}[m^{(a)}_{;i}\, m^{(b)}_{;k} + m^{(a)}_{;k}\, m^{(b)}_{;i} - g_{ik}\, m^{(a);l}\, m^{(b)}_{;l}]\sqrt{(-g)}\,\mathrm{d}^4x \\
&-\frac{\mu}{\lambda}\sum_{a<b}\sum\int_V \delta g^{ik}[g_{ik}\, g^{pq}\{m^{(a)}_{;pq}\, m^{(b)} + m^{(b)}_{;pq}\, m^{(a)} + m^{(a)}_{;p}\, m^{(b)}_{;q} + m^{(a)}_{;q}\, m^{(b)}_{;p}\} \\
&\qquad - \{m^{(a)}_{;ik}\, m^{(b)} + m^{(b)}_{;ik}\, m^{(a)} + m^{(a)}_{;i}\, m^{(b)}_{;k} + m^{(b)}_{;i}\, m^{(a)}_{;k}\}]\sqrt{(-g)}\,\mathrm{d}^4x \\
&+\frac{\mu}{\lambda}\sum_{a<b}\sum\int_V (R_{ik} - \tfrac{1}{2} g_{ik} R)\, m^{(a)} m^{(b)}\sqrt{(-g)}\,\delta g^{ik}\,\mathrm{d}^4x. \qquad (26)
\end{aligned}
$$

The first term can be written as

$$\frac{1}{2}\int_V \delta g^{ik} g_{ip}\, g_{kq}\, \underset{m}{T}^{pq}\sqrt{(-g)}\,\mathrm{d}^4x, \qquad (27)$$

so that $\delta J = 0$ for all such variations gives the result

$$
\begin{aligned}
\frac{2\mu}{\lambda}(R_{ik} - \tfrac{1}{2} g_{ik} R)\sum_{a<b}\sum m^{(a)}\, m^{(b)}
&= -g_{ip}\, g_{kq}\, \underset{m}{T}^{pq} + \frac{2\mu}{\lambda}\sum_{a<b}\sum [m^{(a)}(g_{ik}\, g^{pq} m^{(b)}_{;pq} - m^{(b)}_{;ik}) + m^{(b)}(g_{ik}\, g^{pq} m^{(a)}_{;pq} - m^{(a)}_{;ik})] \\
&\quad + \frac{1}{\lambda}\sum_{a<b}\sum [(1-2\mu)(m^{(a)}_{;i}\, m^{(b)}_{k} + m^{(a)}_{;k}\, m^{(b)}_{;i}) - (1-4\mu)\, g_{ik}\, m^{(a);l}\, m^{(b)}_{;l}]. \qquad (28)
\end{aligned}
$$

These are our field equations.

The present result bears some resemblance to the equations of Jordan (1959) and to those of Brans & Dicke (1961). These theories modify the equations of general relativity. The differences show up, for example in the theory of Brans & Dicke, in the prediction of a variable gravitational constant and a different rate of perihelion rotation of planetary orbits. The present theory, although starting from an outlook radically different from that of general relativity, will be shown to be equivalent to general relativity as far as macroscopic phenomena are concerned.

At this stage we give a reason for the special choice $\mu = \frac{1}{6}$—i.e. for taking the wave equation (5) to be conformally invariant. The contracted scalar form of (28) is

$$-\frac{2\mu}{\lambda} R \sum_{a<b}\sum m^{(a)} m^{(b)} = -T + \frac{6\mu}{\lambda} \sum_{a<b}\sum g^{pq}(m^{(a)} m^{(b)}_{;pq} + m^{(a)}_{;pq} m^{(b)}) + \frac{2}{\lambda}(6\mu - 1) \sum_{a<b}\sum m^{(a);l} m^{(b)}_{;l}. \tag{29}$$

From (3) and (5) we also have

$$g^{pq} m^{(a)}_{;pq} + \mu R m^{(a)} = \lambda \int \delta^{(a)}_{(X,A)} [-\bar{g}(X,A)]^{-\frac{1}{2}} \,\mathrm{d}a. \tag{30}$$

In (29) and (30), $R, T, m^{(a)}, \ldots$ are all taken at a general point x. Using (30) for all the $m^{(a)}$, we can write (29) as

$$\begin{aligned}\frac{2\mu}{\lambda}(6\mu - 1)\Big[R \sum_{a<b}\sum m^{(a)} m^{(b)} - \sum_{a<b}\sum m^{(a);l} m^{(b)}_{;l}\Big] &= -T + 6\mu \sum_{a<b}\sum \Big[m^{(a)} \int \delta^{(4)}_{(X,B)} (-\bar{g})^{-\frac{1}{2}} \,\mathrm{d}b + m^{(b)} \int \delta^{(4)}_{(X,A)} (-\bar{g})^{-\frac{1}{2}} \,\mathrm{d}a\Big] \\ &= -T + 6\mu \sum_a \int m_a \delta^{(4)}_{(X,A)} (-\bar{g})^{-\frac{1}{2}} \,\mathrm{d}a \\ &= (6\mu - 1)T, \end{aligned} \tag{31}$$

in which

$$T = \sum_a \int m_a \delta^{(4)}_{(X,A)} (-\bar{g})^{-\frac{1}{2}} \,\mathrm{d}a$$

from (15) together with $m_a = \sum_{b \neq a} m^{(b)}$. Equation (31) reduces to an identity when $\mu = \frac{1}{6}$, so that this case imposes the least constraint on the mass fields.

In the rest of this paper we shall take $\mu = \frac{1}{6}$. If in future work it should become desirable to change from this value, the necessary alteration in the following formulae is easily made. The field equations now become

$$\begin{aligned}(R_{ik} - \tfrac{1}{2} g_{ik} R)\Big(\sum_{a<b}\sum m^{(a)} m^{(b)}\Big) &= -3\lambda g_{ip} g_{kq} \underset{m}{T^{pq}} \\ &+ \sum_{a<b}\sum [m^{(a)}(g_{ik} g^{pq} m^{(b)}_{;pq} - m^{(b)}_{;ik}) + m^{(b)}(g_{ik} g^{pq} m^{(a)}_{;pq} - m^{(a)}_{;ik})] \\ &+ 2\sum_{a<b}\sum [m^{(a)}_{;i} m^{(b)}_{;k} + m^{(a)}_{;k} m^{(b)}_{;i} - \tfrac{1}{2} g_{ik} m^{(a);l} m^{(b)}_{;l}]. \end{aligned} \tag{32}$$

The Einstein case

A requirement of any new theory of gravitation is that it should meet the classical tests of Einstein's theory, the perihelion motion, the deflexion of light, and the gravitational shift of spectrum lines. We shall demonstrate these results by showing that Einstein's theory can be derived from the present theory in the smooth fluid approximation. Defining the total mass m by

$$m(x) = \sum_a m^{(a)}(x), \tag{33}$$

we approximate to $\sum\sum_{a<b} m^{(a)} m^{(b)}$ by $\frac{1}{2}m^2$. This involves neglecting terms of the type $(m^{(a)})^2$ in comparison with those of type $m^{(a)} m^{(b)}$, $b \neq a$. For n particles there are n terms of the former type and $\frac{1}{2}n(n-1)$ of the latter type, so that when n is very large the approximation is good.

We introduce the proper-density N of particles, defined by

$$N(x) = \sum_a \int \delta^{(4)}_{(X,A)} [-\bar{g}(X,A)]^{-\frac{1}{2}} \, da. \tag{34}$$

Summation of (3) with respect to a then gives

$$g^{pq} m_{;\,pq} + \tfrac{1}{6} R m = \lambda N. \tag{35}$$

In the smooth-fluid approximation N is related to T. Thus

$$T = \sum_a \int m_a \, \delta^{(4)}_{(X,A)} [-\bar{g}(X,A)]^{-\frac{1}{2}} \, da \approx m \sum_a \int \delta^{(4)}_{(X,A)} [-\bar{g}(X,A)]^{-\frac{1}{2}} \, da$$

$$= mN. \tag{36}$$

The approximation involved here is essentially the same as before,

$$m_a = \sum_{b \neq a} m^{(b)} \approx \sum_{b \neq a} m^{(b)} + m^{(a)} = m. \tag{37}$$

We now observe that our whole scheme of equations is solved in the smooth-fluid approximation by

$$m = \text{constant} = m_0, \quad \text{say}, \tag{38}$$

and by Einstein's equations

$$R_{ik} - \tfrac{1}{2} g_{ik} R = -8\pi G \underset{m}{T}_{ik}, \tag{39}$$

but with an explicit formula for the constant G. The complicated terms on the right-hand side of (32) disappear. For example,

$$\sum\sum_{a<b} m^{(a)} g_{ik} g^{pq} m^{(b)}_{;pq} \approx \tfrac{1}{2} g_{ik} g^{pq} m \, m_{;\,pq} = 0. \tag{40}$$

Hence
$$\tfrac{1}{2} m_0^2 (R_{ik} - \tfrac{1}{2} g_{ik} R) = -3\lambda T_{ik}, \tag{41}$$

where for convenience we omit the subscript m on T_{ik} (used to denote that $\underset{m}{T_{ik}}$ was due only to the matter—we are omitting the contributions of fields such as the electromagnetic field). Also from (35),

$$\tfrac{1}{6}Rm_0 = \lambda N \approx \lambda T/m_0, \tag{42}$$

which is the same as the contracted form of (41), so that (41) contains (35). But (41) is the same as Einstein's equations if we write

$$G = \frac{3}{4\pi}\frac{\lambda}{m_0^2}. \tag{43}$$

At this stage we notice that there is nothing in the present theory to prevent us putting the dimensionless λ equal to unity. In fact λ was simply a scale factor introduced into (2) to permit our definition of mass to be adjusted to equal conventional values, if necessary—λ has no physical significance. This is shown immediately by noticing that whereas T_{ik} depends linearly on the mass values (cf. (15)) the left-hand side of (41) depends quadratically. Hence the scale factor λ must be present on the right-hand side. It follows that only one genuinely unknown quantity, m_0, enters the above equations.

The first clear gain over the Einstein theory now emerges, namely that the sign of $R_{ik} - \frac{1}{2}g_{ik}R$ is not arbitrary, it is both determinate and correct. Thus if we adopt the usual convention that masses be positive, i.e. $m_0 > 0$, then $\lambda > 0$ and $G > 0$ from (43). If λ were taken < 0 then $G < 0$. But in this case $m_0 < 0$ and the product $8\pi G T_{ik}$ remains the same, since T_{ik} contains m_0. In the classical Einstein theory gravitation *could* be repulsive, if the opposite sign were chosen for G. In the present theory gravitation must be attractive, a condition known to be necessary if gravitation is viewed from particle physics instead of classically.

It is perhaps desirable to add here a word of caution. The equations describing the theory are linear in mass but non-linear in geometry. This must be borne in mind in considering the smooth-fluid approximation. For example, if we wish to determine the elementary interaction between two particles, we cannot use the g_{ik} *after* the smooth-fluid approximation has been made. Rather, we must go back to the original equations (32). On account of the identity (31), there are nine independent equations in this set. These, together with the equation (38) are to be used to determine g_{ik} and hence $\tilde{G}(A, B)$. Such a calculation, though simple in principle, is extremely difficult to carry through in practice. In the description of macroscopic phenomena, however, we do not need to know the elementary Green functions; the Einstein equations obtained by the smooth-fluid approximation are sufficient for this purpose.

Cosmology

The question immediately follows on from the previous work as to what numerical value m_0 should take. Of course, the value is arbitrary in the sense that m_0 contains the arbitrary constant λ, so our question is really to determine m_0/λ. Since m_0/λ arises from the universe, we expect this to be a cosmological issue.

The Robertson–Walker line element for a homogeneous isotropic universe is

$$ds^2 = dt^2 - S^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \quad (k = 0, \pm 1). \tag{44}$$

The field equations (41) give

$$3\frac{\dot{S}^2+k}{S^2} = \frac{6\lambda}{m_0^2}\rho \quad (\rho = m_0 N), \tag{45}$$

$$2\frac{\ddot{S}}{S}+\frac{\dot{S}^2+k}{S^2} = 0. \tag{46}$$

It is easy to deduce that

$$NS^3 = \text{constant} = N_0 S_0^3 \quad \text{(say)}. \tag{47}$$

These equations can be treated in exactly the same way as in cosmological discussions based on the Einstein theory, with $3\lambda/4\pi m_0^2$ replacing the usual gravitational constant. No expression for m_0 can emerge, however, because the theory contains no universal length—a length can be defined from $S/\dot{S}$ but this changes with the epoch. Thus referring back to (3), the Green function $\tilde{G}$ has dimension (length)$^{-2}$, so that m_0 has dimension (length)$^{-1}$, λ being dimensionless.

A length can be introduced into the theory by adding a suitable second term to the action. Thus in

$$\frac{J}{\lambda} = \sum_{a<b}\sum \iint \tilde{G}(A,B)\,\mathrm{d}a\,\mathrm{d}b + (\lambda f)^{-1}\sum_{a<b}\sum \iint \bar{G}_{;i_A i_B}\,\mathrm{d}a^{i_A}\,\mathrm{d}b^{i_B}, \tag{48}$$

$(\lambda f)^{-1}$ must have dimension (length)^2. Addition of the electromagnetic action

$$-\frac{1}{\lambda}\sum_{a<b}\sum 4\pi e_a e_b \iint \bar{G}_{i_A i_B}\,\mathrm{d}a^{i_A}\,\mathrm{d}b^{i_B} \tag{49}$$

to the right-hand side does not introduce a length, since the double integral in (49) is dimensionless, as also is the first term of (48). As far as we are aware, it is only through the second term of (48) that a cosmological length scale can be introduced. It is natural to choose the length-scale so introduced as our unit of length—i.e. to take $\lambda f = 1$. However, writing $\lambda f = 1$ obscures the dimensionality of all succeeding formulae. We therefore prefer to write $\lambda f = H^2$ where H^{-1} is the new length scale—we use this notation because H will turn out to be the Hubble constant.

The contribution of the first term of (14) to δJ is given in (26). The contribution of the second term is immediately obtained from (12) of (1964*c*). This is

$$-\frac{H^2}{\lambda}\sum_{a<b}\sum \int_V \delta g^{ik}[C_i^{(a)}C_k^{(b)} + C_k^{(a)}C_i^{(b)} - g_{ik}C^{(a)l}C_l^{(b)}]\sqrt{(-g)}\,\mathrm{d}^4X; \quad C_i = \frac{\partial C}{\partial x^i}. \tag{50}$$

In the smooth fluid approximation this becomes

$$-\frac{H^2}{\lambda}\int \delta g^{ik}[C_i C_k - \tfrac{1}{2}g_{ik}C^l C_l]\sqrt{(-g)}\,\mathrm{d}^4x. \tag{51}$$

The field equations are obtained by replacing $\underset{m}{T}^{pq}$ in (32) by

$$\underset{m}{T}^{pq} - \frac{H^2}{\lambda}[C^p C^q - \tfrac{1}{2}g^{pq}C^l C_l]. \tag{52}$$

The smooth fluid field equations are therefore

$$R^{ik} - \tfrac{1}{2}g^{ik}R = -\frac{6\lambda}{m_0^2}\left[\underset{m}{T}^{ik} - \frac{H^2}{\lambda}(C^i C^k - \tfrac{1}{2}g^{ik}C^l C_l)\right], \tag{53}$$

provided a constant mass $m = m_0$ is still possible.

Turning to the mass equation (35) we notice that a solution $m = m_0$ (constant) was possible provided $R \propto \underset{m}{T}$. In the present case, contraction of (53) gives

$$R = \frac{6\lambda}{m_0^2}\left[\underset{m}{T} + \frac{H^2}{\lambda} C^l C_l\right]; \tag{54}$$

Thus $R \propto \underset{m}{T}$ no longer holds.

This apparent difficulty is resolved by taking proper account of the fact that the existence of the C-field implies broken world-lines. In the mass equation

$$g^{pq} m_{;\,pq} + \tfrac{1}{6} Rm = \lambda \sum_a \int \delta^{(4)}_{(X,A)} [-\bar{g}]^{-\frac{1}{2}} \,\mathrm{d}a, \tag{55}$$

the contribution to the right-hand side in a region of space-time containing unbroken world-lines is still λN. To this must be added the contribution from the broken world-lines in that region. Such contribution arises in the form of a C-field term at the end of the world-line. As was shown in (1964*b*), at the end of the world-line of a neutral particle a,

$$m_a \frac{\mathrm{d}a^i}{\mathrm{d}a} = \sum_{b \neq a} C^{(b)\,i}. \tag{56}$$

In the smooth fluid case with constant mass,

$$m_0 \,\mathrm{d}a^i/\mathrm{d}a = C^i, \quad \text{i.e.} \quad C_i C^i = m_0^2. \tag{57}$$

Let there be n such ends in a unit 4-volume—the unit having been defined as above by H^{-1}. Then the total contribution to the right-hand side of (55) can be written as

$$\lambda N + \lambda n = \lambda \frac{T}{m_0} + \lambda n \frac{C_i C^i}{m_0^2}, \tag{58}$$

since at each end $C_i C^i = m_0^2$. Therefore, $m = m_0$ is a solution of (55) provided

$$\tfrac{1}{6} R m_0 = \lambda \left[\frac{T}{m_0} + n \frac{C_i C^i}{m_0^2}\right]. \tag{59}$$

Comparison with (54) shows that

$$n = H^2 m_0/\lambda. \tag{60}$$

Thus, provided the number of ends of world-lines in unit volume is given by (60), m = constant $= m_0$ is a possible solution. Equations (53) then become valid.

In addition to the field equations we also have the source equation for the C-field, given in (1964*b*). In the present treatment all the particles are being given the same weight, and the mass factor m_0 is not absorbed into C. This requires the source equation to be

$$(m_0 H^2/\lambda)\, C^i_{;\,i} = j^i_{;\,i}, \tag{61}$$

$$j^i = m_0 N \,\mathrm{d}x^i/\mathrm{d}s. \tag{62}$$

In the cosmological case, (61), (62) give

$$\frac{H^2}{\lambda}\left(\ddot{C} + 3\frac{\dot{S}}{S}\dot{C}\right) = \dot{N} + 3\frac{\dot{S}}{S} N, \tag{63}$$

which integrates to

$$\dot{C} = \frac{\lambda N}{H^2} + \frac{A}{S^3}, \tag{64}$$

where A is a constant of integration.

Equations (45) and (46) are changed to

$$3\frac{\dot{S}^2+k}{S^2} = \frac{6\lambda}{m_0^2}\left(\rho - \frac{H^2}{2\lambda}\dot{C}^2\right) \quad (\rho = m_0 N), \tag{65}$$

$$2\frac{\ddot{S}}{S} + \frac{\dot{S}^2+k}{S^2} = \frac{3H^2}{m_0^2}\dot{C}^2. \tag{66}$$

With continued expansion the A/S^3 term in (64), and the k/S^2 term in (65) become negligible and the solution tends asymptotically to a situation in which

$$\dot{C} = m_0, \quad S = e^{Ht}, \quad N = m_0 H^2/\lambda = N_0 \quad \text{(say)}. \tag{67}$$

A comparison with (60) shows that $N_0 = n$. That is, in the asymptotic steady state the number of new particles created in a unit four-dimensional proper volume is equal to the number of particles already present in that volume. The density of matter is given by

$$\rho = m_0 N_0 = m_0^2 H^2/\lambda. \tag{68}$$

Using the result (43) this becomes

$$\rho = 3H^2/4\pi G, \tag{69}$$

a result in agreement with the field formulation of the theory (1964*a*).

The expression of mass and creation in terms of direct particle fields is so similar to that of the electromagnetic field that the question arises whether there is an absorber theory of 'm' and 'C' fields on the lines developed by Wheeler & Feynman (1945) for electromagnetism. In principle such a theory would exist; but it would be much more complicated than the absorber theory of electromagnetic radiation. In the electromagnetic case, as was pointed out in an earlier paper (Hoyle & Narlikar 1964*a*), the effect of electromagnetism on gravitation is ignorable. This makes it possible to superpose linearly the elecromagnetic contributions from individual particles. The m- and C- field energy tensors, however, make a significant contribution to gravitation—they play a vital role in determining g_{ik}. It is therefore impossible to decouple gravitation from the absorber theory, i.e., to superpose contributions to the radiative reaction from individual particles linearly. This is another feature of the non-linearity of this theory mentioned in the last section.

In local problems, where the local proper density N is very large compared to the cosmological density N_0, the C-field terms in (53) may be neglected. Then, writing the matter tensor in the form

$$\underset{m}{T^{ik}} = m_0 N \frac{dx^i}{ds}\frac{dx^k}{ds}, \tag{70}$$

which is the conventional expression, with $c = 1$ and with the fluid motion given by dx^i/ds, we have

$$R^{ik} - \tfrac{1}{2}g^{ik}R = -\frac{6\lambda}{m_0} N \frac{dx^i}{ds} dx = -6H^2 \frac{N}{N_0}\frac{dx^i}{ds}\frac{dx^k}{ds}. \tag{71}$$

The factor H^2 appears on the right-hand side because the dimension of $R^{ik} - \frac{1}{2}S^{ik}R$ is $(\text{length})^{-2}$. If H is put unity, local curvature effects will be determined relative to the inverse of the Hubble constant. The meaning of (71) is that local gravitational effects are determined by the number of particles in the locality, not simply in the sense of the Newtonian and Einstein theories, but by the ratio this number bears to the cosmological number N_0.

The difference is vividly shown by considering what would happen to the solar system if the rest of the universe were removed. In Newtonian theory, nothing. In Einstein's theory also nothing, with a proviso about boundary conditions at infinity—i.e. that space-time be flat at infinity. In the present theory such a hypothetical proposition cannot be contemplated at all. Even a change in N_0 by a factor 2 would grossly change the properties of the Sun. In the case of a reduction, local gravitation is made stronger, space-time curvature is increased. Greater internal pressure would be required inside the Sun in order to maintain mechanical equilibrium, and the consequent increase of the solar energy flux would be large, some 5 magnitudes or more. Take away a half the distant parts of the universe and the Earth would be fried to a crisp.

Much more is involved here than the result itself. If the present line of argument is at all correct it will mean abandoning the conventional point of view of physics, that the local behaviour of matter can be entirely determined by local experiments; by local behaviour we mean all physically possible forms of behaviour—e.g. the increase of solar luminosity—not just the behaviour actually observed. The latter is restricted by the obvious fact that we cannot experiment with the universe. All we determine by experiment is a subset of all possible forms of behaviour.

This is a convenient moment to mention a feature of the field equation that seems to us of logical importance. In the absence of particles, indeed unless there are two or more particles, the field equations reduce to the identity $0 = 0$. The equations $R_{ik} = 0$ for an empty world do not arise. It is worth noticing the curious way in which these equations emerge in the Einstein theory. Start with the conventional action,

$$\frac{1}{16\pi G}\int R\sqrt{(-g)}\,\mathrm{d}^4x - \Sigma m\int \mathrm{d}s \tag{72}$$

leading to

$$R^{ik} - \tfrac{1}{2}g_{ik}R = -8\pi G \underset{m}{T^{ik}}. \tag{73}$$

Now remove the particles, $\underset{m}{T^{ik}} = 0$. Contraction of (73) then gives $R = 0$, so that, finally, $R^{ik} = 0$. But if there are no particles, and $R = 0$, the action (72) is zero, so that the equations $R^{ik} = 0$ have been produced out of nothing. On the other hand, we cannot know that $R = 0$ until we have obtained (73), and having reached (73) we end with $R^{ik} = 0$. There is an element of Russell's 'barber' paradox in this. In the present theory removing the particles corresponds to what in computer terminology would be described as 'clearing store'.

After emphasizing these attractive features of the theory, we return to the failure of our attempt to determine m_0/λ. The equation

$$m_0/\lambda = N_0 H^{-2} \tag{74}$$

implies that m_0/λ would be known if N_0 could be found. This does not seem to be within the scope of the above theory, which therefore must be judged incomplete. We are not seriously perturbed by this failure, however, because a worrying situation would arise if indeed the theory were complete.

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So far it has not been necessary to consider the whole of the action,

$$J = \sum_{a<b}\sum \lambda \iint \tilde{G}(A,B)\,\mathrm{d}a\,\mathrm{d}b + \sum_{a<b}\sum \lambda H^{-2} \iint \bar{G}_{;\,i_A\,i_B}\,\mathrm{d}a^{i_A}\,\mathrm{d}b^{i_B} - \sum_{a<b}\sum 4\pi e_a e_b \iint \bar{G}_{i_A\,i_B}\,\mathrm{d}a^{i_A}\,\mathrm{d}b^{i_B}, \tag{75}$$

since the electromagnetic contribution is not important in the usual formulation of cosmology. But the electromagnetic term must be included to deal with local phenomena. If we take account of $e_a = \pm e$, with e fixed and > 0, $4\pi\,|e_a e_b| = 4\pi e^2$ for all pairs of particles. Now this coefficient cannot have any dimensionality, and for the combination (75) to have plausibility the coefficient should be unity, or at any rate a number of order unity. Comparison with observation, using the equation of motion (8), requires $4\pi e^2 \approx 10^{36}$ if the 'particles' under consideration are protons. For electrons, the corresponding result is $4\pi e^2 \approx 10^{39}$. (It will be recalled that we have been concerned so far with identical particles. The empirical result is therefore different according as we use observational evidence for the inertial mass of the proton or of the electron. The theory obviously requires generalization to deal with particles of different kinds, but this incompleteness seems much less important than that at present under discussion.) We are simply restating the well-known fact that the gravitational force between a pair of charged 'elementary particles' is less in magnitude than the electrical force by a factor of the general order of 10^{40}; 10^{36} for a pair of protons, 3×10^{39} for an electron and proton, 10^{43} for a pair of electrons. If the theory were 'complete' we would be forced to accept the introduction of a dimensionless number of this order, obviously an absurdity.

The 'coincidence' that, empirically, $N_0^{\frac{1}{2}} \approx 10^{40}$, emphasizes the absurdity. We cannot, however, introduce $N_0^{\frac{1}{2}}$ as a multiplying constant in place of $4\pi e^2$ because $N_0^{\frac{1}{2}}$ possesses dimensionality. The implication seems to us to be that the conjunction of the first and third terms in (75) is wrong, just as we believe the conjunction of the two terms of (72) to be wrong. Just as we believe (72) to be an approximate separation of the first term of (75), so we suspect that this term itself, together with the third term, is a separation of some more fundamental expression. The aim would be to derive an equation of the type $H^3 e^4 \approx N_0$, in analogy with the result we have already obtained, $H^2 m_0/\lambda = N_0$. We hope to deal with this problem in future work.

To conclude the present section, we return to the trivial issue of the scale factor λ. We carried this factor throughout the above work, in spite of there being no physical significance in doing so, because $H^2 m_0 = N_0$ for $\lambda = 1$ would have had a strange look. Since N_0 is empirically of order 10^{80}, the mass associated with each of the particles would be $\sim 10^{80}$. It is easy to see how this result arises. We have, with $\lambda = 1$,

$$m_0 = \sum_a m^{(a)}(X) = -\sum_a \int \tilde{G}(X,A)\,\mathrm{d}a. \tag{76}$$

The effective contribution to the sum comes from particles at less than unit distance —i.e. less than H^{-1}—the contributions of more distant particles being 'cut off', analogously to the red-shift of light. On the average there is unit contribution from each of these particles, and there are N_0 of them.

A convenient choice for λ is $\lambda = N_0^{-2}H^6$, where we keep the factor in H to emphasize that λ is dimensionless—we are considering H as unity throughout. Then

$$H^2 m_0 = 1/N_0,$$

so that the sum of the masses of all particles in unit volume is unity.

Concluding remarks

By working with the smooth fluid approximation we have avoided the challenging problem of the form of the gravitational field near a world-line. The simplest statement of the problem occurs for a particle at rest in the usual cosmological co-ordinates—those used in the line element (44).

Assuming the universe (i.e. other particles) to be homogeneously and isotropically distributed one would expect to be able to write the line element near the chosen particle in the form

$$ds^2 = e^\nu \, dT^2 - e^\lambda \, dR^2 - R^2(d\theta^2 + \sin^2\theta \, d\phi^2), \tag{77}$$

in which the co-ordinates T, R are not the same as the t, r of (44). Because of the singularity at the particle, (77) is not expected to be transformable into (44). If the universe is 'steady', it might be expected that the line element (77) would be static. The problem is then to determine a solution for λ, ν, using the field equations (32), that can be carried without physical contradiction to $R = 0$.

This problem has not been solved in the usual theory. The best that can be done is to solve the field equations outside a tube surrounding the particle, subject to chosen boundary conditions on the tube. An attempt to carry the solution to $R = 0$ encounters the difficulty that the situation is analogous to the macroscopic case of a spherically symmetric body, with

$$e^\nu = e^{-\lambda} = 1 - \text{constant}/R, \tag{78}$$

and there is an unacceptable singularity as $R \to 0$ unless the particle be taken to possess finite extension—i.e. unless the particle becomes a tube.

There was the immediate hope that, the field equations being different in the present theory, the singularity might be avoided. The singularity arises from the δ-function in $\underset{m}{T^{ik}}$, which is the same in the present theory as in the usual theory. But in the present theory there are non-zero contributions to the Ricci tensor *outside* the singularity—the right-hand side of (32) is not confined to a δ-function at the world-line of a particle. Investigation of this problem within the purely gravitational theory has turned up a curiously ironic situation. The usual δ-function terms seem to give no trouble but the new non-zero terms outside the singularity are difficult to control.

Near the world-line a it is to be expected that $m^{(a)}$ behaves as $1/R$, due to the δ-function at the world-line. Sufficiently near the world-line, for small enough R that is, the summation $\sum\sum_{a<b} m^{(a)} m^{(b)}$ on the left-hand side of (32) behaves as $1/R$. This factor introduces a convergence in $(R_{ik} - \frac{1}{2} s_{ik} R)$, sufficient to control the δ-function on the right-hand side of (32), but apparently not sufficient to control the $m_{;ik}$ term.

We hope to deal with this problem in a future paper. For the present, our opinion to date is that the singularity problem cannot be solved within a purely gravitational theory. It seems that terms due to one field must cancel those due to another—in particular that the $m_{;ik}$ term in (32) must cancel with another field, perhaps the electromagnetic field. Our impression is that a special condition will have to be satisfied for this to be the case, and that this condition may turn out to lead into the problem discussed at the end of the preceding section.

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