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MACH'S PRINCIPLE IN GENERAL RELATIVITY

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SUMMARY

Mach's Principle is taken as a criterion for selecting cosmological solutions of the Einstein field equations, in which, in a well-defined manner, the metric arises from material sources alone. In such model universes inertial forces are due to the gravitational interaction of matter, and there is a relativity of accelerated motion. The problem of stating such a selection rule in general relativity divides into two parts: an analysis of the relation of the metric to the Riemann curvature, and of the curvature to the stress tensor, with associated Machian criteria. From the first criterion we show that Mach's Principle is not satisfied in Minkowski space. It seems that asymptotically flat space-times are also non-Machian, as required by the Machian philosophy. The second criterion rules out vacuum solutions and spatially homogeneous cosmological models containing perfect fluids in which there is anisotropic expansion or rotation. Mach's Principle is found to be satisfied in Robertson–Walker models and in a simple class of inhomogeneous solutions. These results lead us to suggest that Mach's Principle may play a role in explaining the observed gross features of the Universe.

1. INTRODUCTION

Einstein gave the name ' Mach's Principle ' to the following related ideas: that only relative motion is observable, and hence that there should be no dynamically privileged reference frames; that inertial forces should arise from a gravitational interaction between matter only, and so from an observer-dependent splitting of the total gravitational field; that space-time is not an absolute element of physics, but that its metric structure is totally dependent on the matter content of the Universe.

These ideas are incorporated in general relativity (covariance, equivalence principle, field equations), but in an incomplete way, as evidenced, for example, by the empty space-time solutions of the field equations, in which there are inertial forces relative to no matter. In order to incorporate Mach's Principle fully into the theory, it is natural to add a criterion for selecting as physically admissible *global* solutions, only those general relativistic cosmological models in which Mach's Principle is satisfied. The purpose of this paper is to show that it is possible to set up such a selection rule. As a consequence we shall be able to rule out asymptotically-flat space-times and empty space-times as possible cosmological models. Amongst cosmologies with homogeneous spatial sections it will be found that only the Robertson–Walker models satisfy the Mach conditions. Our formulation is therefore in agreement with intuitive notions of Mach's Principle in excluding the possibility of absolute rotation in homogeneous cosmologies. Finally, we shall suggest that Mach's Principle may play a fundamental role in providing an explanation of the large-scale homogeneity and isotropy of the Universe.

The possibility of less than total dependence of the metric, and hence inertial forces, on the matter content, arises through the boundary conditions to be imposed on the field equations. The contribution of boundary values and sources to a solution of a system of partial differential equations is most clearly displayed in an integral representation. Following work of Al'tshuler (1967) and Lynden-Bell (1967), Sciama, Waylen & Gilman (1969) were able to show that an integral representation of the Einstein field equations could be obtained in the form

$$g^{\alpha'\beta'}(x') = -2\kappa \int_V G^{\alpha'\beta'}_{\mu\nu}(x', x)(T^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}T)\sqrt{-g}\, d^4x + \int_{\partial V} G^{\alpha'\beta'}_{\mu\nu;\rho} g^{\mu\nu}\sqrt{-g}\, dS^\rho,$$

in the case of zero cosmological constant. We call this the SWG integral representation. From it one sees how the metric of the Riemannian space-time (M, g) at the point $P'(x')$ is constructed out of contributions from the energy-momentum tensor at points $P(x)$ in a normal neighbourhood V of P', and from the values of the metric on the boundary ∂V. These contributions are propagated through the space-time by the retarded Green ' function ', $G^{\alpha'\beta'}_{\mu\nu}(x', x)$, for the self-adjoint tensor wave equation

$$\Box\, \phi^{\mu\nu} + 2R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}\phi^{\rho\sigma} = -2\kappa K^{\mu\nu}.$$

This equation, and the gauge condition

$$(\phi^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}g_{\rho\sigma}\phi^{\rho\sigma})_{;\mu} = 0$$

constitute the differential equations of the SWG theory. In the limit $\phi^{\mu\nu} \to g^{\mu\nu}$, $K^{\mu\nu} \to T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T$, we recover the Einstein field equations, and the gauge condition is automatically satisfied. Unlike the truly linear theory, the Green function here depends on the metric in the representation of which it occurs. However, the differential equations are constructed in such a way that the integral representation possesses a certain stability property: a perturbation is propagated through the space-time by the unperturbed Green function. This property makes it meaningful to speak of the superposition of the contributions to the gravitational potential from the matter fields in the given space-time, despite the non-linearity of the underlying theory. The integral representation would therefore appear to provide a suitable starting point for the imposition of Machian boundary conditions.

In a globally hyperbolic space-time, V may be extended from a normal neighbourhood to the whole manifold (Choquet-Bruhat 1968). A natural Mach condition would then seem to be that in the limit $V \to M$, the boundary integral should vanish (Gilman 1970). This rules out the possibility of a source-free contribution to the metric. However, the coordinate invariance of the field equations of general relativity, or of the SWG theory, implies the existence of constraints on the initial data. In a Universe with particle horizons, the matter outside a horizon may contribute to the potential at the field point through the constraint equations on a global Cauchy surface containing the field point (Penrose 1964). This contribution could not be contained in the completely causal four-volume integral. In such a case, to demand the vanishing of the surface integral would involve the elimination of contributions from matter. While such a demand is not self-inconsistent, we believe it to be too strong a condition, not in the spirit of the original idea.

The correct statement of the Mach criterion along these lines would involve the separation of the constraint variables from the gauge variables and true dynamical degrees of freedom in the metric. This problem has never been completely solved: we therefore seek a way of avoiding a direct approach. Instead, we divide the problem into two parts. The first involves the relation of the metric to the Riemann tensor. Whereas the vanishing of $T_{\mu\nu}$ in a region tells us little about the metric there, the vanishing of the curvature means that space-time is flat. There is sufficient locality in this situation that the problem of co-ordinate freedom can be dealt with. The second step then relates the Riemann tensor to the matter content. Here, there is no problem of gauge freedom since the curvature is physically observable, and it turns out that one is left with a soluble constraint problem. At each stage we are presented with a system of differential equations and a known solution. For a solution to satisfy Mach's Principle, we require that it be possible to represent it in a particular, manifestly Machian, way. The two conditions which arise therefore provide a natural statement of Mach's Principle as a selection rule.

2. NOTATION

In this section we collect the notation to be used, and establish our sign conventions.

By a Riemannian manifold, we mean a differentiable manifold with a Lorentz metric having signature +2. Greek indices will denote coordinate components and range over 0, 1, 2, 3. Latin indices from the first part of the alphabet are used for tetrad components and have the range 0 to 3; from the latter part of the alphabet $(i, j, \ldots)$, they range over 1, 2, 3, and denote either coordinate or tetrad components as the context requires. The summation convention is used throughout.

Covariant differentiation is denoted by the operator ∇_μ, or by a semicolon, and partial differentiation by ∂_μ or a comma.

The Riemann tensor is defined by the Ricci identity

$$u^\mu{}_{;\rho\sigma} - u^\mu{}_{;\sigma\rho} = -R^\mu{}_{\lambda\rho\sigma}u^\lambda$$

and is

$$R^\lambda{}_{\mu\rho\sigma} = -\partial_\sigma\Gamma^\lambda{}_{\mu\rho} + \partial_\rho\Gamma^\lambda{}_{\mu\sigma} - \Gamma^\lambda{}_{\sigma\nu}\Gamma^\nu{}_{\mu\rho} + \Gamma^\lambda{}_{\rho\nu}\Gamma^\nu{}_{\mu\sigma}.$$

The Ricci tensor is $R_{\mu\sigma} = R^\lambda{}_{\mu\lambda\sigma}$ and the Einstein field equations are

$$R^\mu{}_\nu - \tfrac{1}{2}\delta^\mu_\nu R = \kappa T^\mu{}_\nu.$$

Round brackets denote symmetrization, $u_{(\mu\nu)} = \frac{1}{2}(u_{\mu\nu}+u_{\nu\mu})$, and square brackets antisymmetrization, with the convention that

$$u_{[\mu[\nu\rho]\sigma]} = \tfrac{1}{4}(u_{\mu\nu\rho\sigma} - u_{\rho\nu\mu\sigma} + u_{\rho\sigma\mu\nu} - u_{\mu\sigma\rho\nu});$$

that is, antisymmetrization over $[\mu, \rho]$ and $[\nu, \sigma]$.

The operator $\Box$ will always denote the covariant d'Alembertian, $g^{\mu\nu}\nabla_\mu\nabla_\nu$, and the symbol $\underset{\xi}{\mathcal{L}}$ will denote the Lie derivative with respect to the vector field ξ.

The permutation symbols, $\epsilon^{\alpha\beta\gamma\delta}$, $\epsilon_{\alpha\beta\gamma\delta}$, satisfy, by definition,

$$\epsilon^{0123} = +1 = \epsilon_{0123}.$$

We define a permutation tensor, $\eta_{\alpha\beta\gamma\delta}$, by

$$\eta_{\alpha\beta\gamma\delta} = (-g)^{1/2}\,\epsilon_{\alpha\beta\gamma\delta}, \quad g = \det(g_{\mu\nu}),$$

and obtain the contravariant components by raising indices with the metric tensor; so

$$\eta^{\alpha\beta\gamma\delta} = g^{\alpha\lambda}g^{\beta\mu}g^{\gamma\nu}g^{\delta\rho}\eta_{\lambda\mu\nu\rho} = -(-g)^{1/2}\,\epsilon^{\alpha\beta\gamma\delta}.$$

In gaussian coordinates, we define

$$\eta^{ijk} = -\eta^{0ijk} = (h)^{-1/2}\,\epsilon^{ijk}$$

$$\eta_{ijk} = \eta_{0ijk} = (h)^{1/2}\,\epsilon_{ijk},$$

where h_{ij} is the positive definite 3-space metric, and ϵ_{ijk} the three-dimensional permutation symbol.

The (left) dual of an antisymmetric tensor is defined by

$$*F_{\alpha\beta} = \tfrac{1}{2}\eta_{\alpha\beta\gamma\delta}F^{\gamma\delta}.$$

This establishes a sign convention.

The Weyl tensor is

$$C_{\lambda\mu\nu\rho} = R_{\lambda\mu\nu\rho} - 2g_{[\lambda[\nu}R_{\mu]\rho]} + \tfrac{1}{3}g_{\lambda[\nu}g_{\rho]\mu}R.$$

In a gaussian coordinate system, or in tetrad components, we put

$$E_{ij} = C_{0i0j}, \quad B_{ij} = -\tfrac{1}{2}\eta_{klj}C_{0i}{}^{kl} = -{*C^{0i}}_{0j},$$

and construct the complex 3-tensor:

$$V_{ij} = E_{ij} + iB_{ij}.$$

We use units in which $c = 1$ and where convenient we also put $\kappa = 1$.

3. THE GENERALIZED SWG EQUATION

In order to impose a Mach condition on the metric-curvature relation, we shall need an analogue of the SWG integral representation with the Riemann tensor as the source term. This will enable us to see how the effects of curvature are superimposed to generate a metric. It is not sufficient to quote the result (Hlavaty 1960) that in general the Riemann tensor uniquely determines a metric, since such a statement is not manifestly Machian.

We start from the definition

$$R^{\lambda\mu}{}_{\rho\sigma} = g^{\nu\mu}R^{\lambda}{}_{\nu\rho\sigma} \tag{1}$$

in a given Riemannian manifold. On making an arbitrary variation in the contravariant metric $g^{\mu\nu} \to g^{\mu\nu} + \epsilon h^{\mu\nu}$, where ϵ is a small parameter, the Riemann tensor becomes, to first order in ϵ

$$R^{\lambda\mu}{}_{\rho\sigma} + \epsilon \overset{(1)}{R}{}^{\lambda\mu}{}_{\rho\sigma} = g^{\nu\mu}R^{\lambda}{}_{\nu\rho\sigma} + \epsilon h^{\mu\nu}R^{\lambda}{}_{\nu\rho\sigma} + g^{\nu\mu}\overset{(1)}{R}{}^{\lambda}{}_{\nu\rho\sigma}. \tag{2}$$

Defining

$$K_{\lambda\mu\rho\sigma} = (R^{\alpha\beta}{}_{\rho\sigma} + \epsilon\overset{(1)}{R}{}^{\alpha\beta}{}_{\rho\sigma})\,g_{\alpha\lambda}g_{\beta\mu} + (R^{\alpha\beta}{}_{\lambda\mu} + \epsilon\overset{(1)}{R}{}^{\alpha\beta}{}_{\lambda\mu})\,g_{\alpha\rho}g_{\beta\sigma}$$

$$\phi^{\mu\nu} = g^{\mu\nu} + \epsilon h^{\mu\nu},$$

we can use the linearity of $\overset{(1)}{R}{}^{\lambda}{}_{\nu\rho\sigma}$ in $\phi^{\mu\nu}$ to obtain (2) in the form

$$\phi_{[\lambda[\rho;\mu]\sigma]} + \phi_{[\rho[\lambda;\sigma]\mu]} + \tfrac{1}{2}\phi_{\nu[\lambda}R^{\nu}{}_{\mu]\rho\sigma} + \tfrac{1}{2}\phi_{\nu[\rho}R^{\nu}{}_{\sigma]\lambda\mu} = \tfrac{1}{2}K_{\lambda\mu\rho\sigma} \tag{3}$$

where the antisymmetrization in $\phi_{[\lambda[\rho;\mu]\sigma]}$ is over the pairs of indices $[\lambda\mu]$, $[\rho, \sigma]$. We call (3) the generalized SWG equation. From the manner of its derivation (3) is an identity, but we can regard it now as an equation for $\phi^{\mu\nu}$ in terms of a known $K_{\lambda\mu\rho\sigma}$, with at least one known solution, namely $g^{\mu\nu}+\epsilon h^{\mu\nu}$, in the given background.

On taking the trace of (3), one recovers the SWG equations in a general gauge. Other forms of (3) could be obtained by choosing the covariant metric or a tensor density as the basic variable, but these would not yield the SWG equations on contraction. However, the results in this paper (examples 1, 2, 3, proposition 4) are independent of the precise choice of the form of (3).

One obtains a more symmetrical system, and a closer analogy with electrodynamics, on introducing a 'superpotential' $Z^{\lambda}{}_{\mu\rho\sigma}$, related to $\phi_{\mu\sigma}$ by

$$\phi_{\mu\sigma} = Z^{\lambda}{}_{\mu\lambda\sigma}, \tag{4}$$

and required to have the algebraic symmetries of the Riemann tensor. This superpotential is then to be compared with the Lorentz invariant bivector of the Hertz potential in electrodynamics. We can now write the generalized SWG equation in the symbolic form

$$LZ = K. \tag{5}$$

Since we have at least the freedom to make coordinate variations of order ϵ, the generalized SWG equations must be invariant to this order under the transformations

$$\begin{aligned} \phi_{\mu\nu} &\to \phi_{\mu\nu} + 2\epsilon\xi_{(\mu;\nu)} \\ K^{\lambda\mu}{}_{\rho\sigma} &\to K^{\lambda\mu}{}_{\rho\sigma} + \underset{\epsilon\xi}{\mathscr{L}} R^{\lambda\mu}{}_{\rho\sigma} \end{aligned}$$

with arbitrary ξ_{μ}. In terms of $Z^{\lambda}{}_{\mu\nu\rho}$, we have the gauge freedom

$$\begin{aligned} Z_{\lambda\mu\rho\sigma} &\to Z_{\lambda\mu\rho\sigma} + 2g_{[\lambda[\rho}\Lambda_{\mu]\sigma]} + \epsilon\epsilon_{\lambda\mu\rho\sigma} \\ \Lambda_{\mu\nu} &= 2\epsilon\xi_{(\mu;\nu)} - \tfrac{1}{3}g_{\mu\nu}\epsilon\xi^{\lambda}{}_{;\lambda} \\ K^{\lambda\mu}{}_{\rho\sigma} &\to K^{\lambda\mu}{}_{\rho\sigma} + \underset{\epsilon\xi}{\mathscr{L}} R^{\lambda\mu}{}_{\rho\sigma} \end{aligned} \tag{6}$$

where $\epsilon_{\lambda\mu\rho\sigma}$ is required to have the symmetries of the Weyl tensor, but is otherwise arbitrary. This freedom may be used to impose a gauge condition.

Proposition 1. $Z^{\lambda}{}_{\mu\nu\rho}$ may be restricted by imposing the 'Bianchi gauge conditions'

$$Z^{\lambda}{}_{\mu[\nu\rho;\sigma]} = 0. \tag{7}$$

Proof. Suppose $\bar{Z}^{\lambda}{}_{\mu\nu\rho}$ satisfies the generalized SWG equations but not the gauge conditions (7). We show that (7) may be satisfied by a transformation of the form (6). Indeed, let $Z_{\lambda\mu\rho\sigma} = \bar{Z}_{\lambda\mu\rho\sigma} + 2g_{[\lambda[\rho}\Lambda_{\mu]\sigma]} + \epsilon\epsilon_{\lambda\mu\rho\sigma}$ as in (6), and let $\bar{E}_{\lambda\mu\rho\sigma}$ be the trace-free part of $\bar{Z}_{\lambda\mu\rho\sigma}$. Then, in terms of the right dual, $\bar{E}^{*}{}_{\lambda\mu\rho\sigma}$, the Bianchi conditions read

$$\left.\begin{aligned} \nabla^{\alpha}\epsilon^{*}{}_{\lambda\mu\rho\alpha} &= -\nabla^{\alpha}\bar{E}^{*}{}_{\lambda\mu\rho\alpha} + s_{\lambda\mu\rho} \\ s^{\alpha}{}_{\mu\alpha} &= 0. \end{aligned}\right\} \tag{8}$$

The second set of equations are inhomogeneous wave equations for ξ_{μ}, and determine the transformation of $\bar{\phi}_{\mu\nu}$ to the SWG gauge. Differentiation of the first set then leads to inhomogeneous wave equations for $\epsilon_{\lambda\mu\rho\sigma}$ which can be solved subject to the initial constraints imposed by (8).

Note that the gauge conditions of the SWG theory $(\phi_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\phi)^{;\nu} = 0$, are obviously the contracted Bianchi conditions. By algebraic manipulation of (3), (4), (7), using the Ricci identities, one obtains:

Proposition 2. In the Bianchi gauge, the generalized SWG equation takes the form

$$\Box Z_{\lambda\mu\rho\sigma}+g_{\alpha\gamma}g_{\beta\delta}R^{\alpha}{}_{[\lambda}{}^{\beta}{}_{[\rho}Z^{\gamma}{}_{\mu]}{}^{\delta}{}_{\sigma]}+\tfrac{1}{2}(R^{\beta}{}_{[\lambda}Z_{\mu]\beta\rho\sigma}+R^{\beta}{}_{[\rho}Z_{\sigma]\beta\lambda\mu} - Z^{\beta}{}_{[\lambda}R_{\mu]\beta\rho\sigma}-Z^{\beta}{}_{[\rho}R_{\sigma]\beta\lambda\mu}) = 2K_{\lambda\mu\rho\sigma}. \quad (9)$$

In particular, in Minkowski space-time, we have

$$\Box Z_{\lambda\mu\rho\sigma} = 2K_{\lambda\mu\rho\sigma}; \qquad \Box \equiv \eta^{\alpha\beta}\,\partial_\alpha\,\partial_\beta. \quad (10)$$

Proposition 1 implies the compatibility of the Bianchi gauge conditions and the wave equations (9). From (10), it is clear that in flat space-time the compatibility conditions are the Bianchi identities for $K_{\lambda\mu\rho\sigma}$. In general, one can show that:

Proposition 3. To first order in ϵ, the compatibility conditions for (7) and (9) are the varied Bianchi identities.

The generalized SWG equation expresses the Riemann tensor in terms of a linear operation on the metric (in a given background). In order to impose a Mach condition we need to discover when this relation can be inverted to express the metric as a linear functional (in the mathematical sense) of the Riemann tensor. The standard procedure would be to construct a Green function for the non-singular system (9), using the gauge conditions (7) to impose constraints on the initial data This we wish to avoid, since we cannot solve the constraints explicitly. It is possible to obtain an inverted form of (5) directly, by using the concept of a generalized inverse (Penrose 1954; Petryshyn 1967).

Consider the system of linear equations in $\mathbb{R}^n$

$$M\mathbf{z} = \mathbf{k}, \quad \det M = 0, \quad \mathbf{k}\epsilon(\ker M)^{\perp} = \mathscr{R}(M). \quad (11)$$

A particular solution can be constructed by means of a generalized inverse matrix, M^+, defined by

$$MM^+M = M, \quad M^+MM^+ = M^+, \quad (12)$$

since $\mathbf{z} = M^+\mathbf{k}$ is then a solution. If M is symmetric an orthogonal transformation will bring it to the form diag$(\lambda_1 \ldots \lambda_p, 0 \ldots 0)$, and the matrix diag$(\lambda_1^{-1} \ldots \lambda_p^{-1}, 0 \ldots 0)$, referred to the original basis, is a generalized inverse of M. If M is not symmetric we premultiply by its transpose, $M^\top$, and diagonalize $M^\top M$ to obtain $M^+ = (M^\top M)^+ M^\top$. Equations (12) do not determine M^+ uniquely. For any generalized inverse, the general solution of (11) is

$$\mathbf{z} = M^+\mathbf{k}+(\mathbb{1}-M^+M)\,\mathbf{y} \quad (13)$$

where $\mathbf{y}$ is arbitrary.

A generalized inverse is essentially an extension to the whole of $\mathbb{R}^n$ of the inverse of the non-singular map obtained by restricting the domain of M to the orthogonal complement of its kernel. One therefore expects the procedure to work for mappings between closed subspaces of Hilbert spaces (Desoer & Whalen 1963). For the generalized SWG system, we want to regard (5) as the analogue of (11), for an appropriate choice of Hilbert spaces. The analogy is not quite complete, since the gauge arbitrariness in Z, given by (6), involves a transformation of K. To restore the analogy, we can form equivalence classes of K's under gauge transformations as in Appendix 1. Alternatively, we can drop the arbitrary term in the solution of (13) in the general case. We shall show it can reappear for special background metrics. In either case the non-singular form of the system, equations (7) and (9), is used to generate an inverse on closed subspaces, and this inverse is extended to a generalized inverse satisfying (12).

Theorem 1. Let U be a relatively compact open subset of a given Riemannian manifold, and consider the generalized SWG equation $LZ = K$ in U. Then there exists a generalized inverse operator L^+ to the singular differential operator L.

In Appendix 1 we elaborate the arguments leading to this result.

4. THE FIRST MACH CONDITION

Using the results of the preceding section, we can write a solution to the generalized SWG equation (3) in the form

$$\phi = trL^+[K]. \tag{14}$$

If we are regarding K as a class of varied Riemann tensors, equivalent under gauge transformations, then ϕ is arbitrary up to the addition of a gauge term. Otherwise, there is no arbitrariness in ϕ, given K, but many ϕ's will be related by a gauge transformation. In general, if we now take the limit $\epsilon \to 0$ in (14), we obtain a representation of the given metric, $g^{\mu\nu}$, as a linear functional $L^+[R]$ of the Riemann tensor. From the manner of its derivation, this representation has the same stability property as that of the SWG theory (Section 1). It gives us a meaningful way of stating that, in a given space-time, the metric is ' due to ' the curvature; that is, the metric, and hence inertial forces, are generated by physical sources (curvature) only. This is the generic situation and it is manifestly Machian.

Now, this is certainly the situation if there is no arbitrariness in the limit $\epsilon \to 0$, which will be the case if the arbitrariness in ϕ is of order ϵ, or if all ϕ's related by gauge transformations represent variations away from the same background metric. However, it may happen, for particular metrics, that there exists a vector field ξ such that $\underset{\xi}{\mathscr{L}} R^{\lambda\mu}{}_{\rho\sigma} = O(\epsilon)$ or, equivalently, $\underset{\epsilon\xi}{\mathscr{L}} R^{\lambda\mu}{}_{\rho\sigma} = O(\epsilon^2)$. We are then free to choose whether or not we add $\underset{\xi}{\mathscr{L}} g^{\mu\nu} = 2\nabla^{(\mu}\xi^{\nu)} = O(1)$ to $\phi^{\mu\nu}$ in (14). The limit $\epsilon \to 0$ will then not yield the background metric for at least one choice. If the solution of the generalized SWG equation is not unique for some given curvature (up to choice of coordinates), it no longer makes sense to say that a solution is generated by physical sources only. In these circumstances it would not be sensible to regard the background metric as a linear functional of the curvature. Such a situation is manifestly non-Machian.

With this distinction in mind, we summarize:

The first Mach condition. We shall say a space-time satisfies the first Mach condition if the metric is locally a generalized inverse functional of the Riemann tensor.

Some examples should make this clear.

Example 1. *Minkowski space-time*. In this case $\underset{\xi}{\mathscr{L}} R^{\lambda\mu}{}_{\rho\sigma} = 0$ for an arbitrary ξ^μ since $R^{\lambda\mu}{}_{\rho\sigma} = 0$. We expect $L^+[R] = 0$ so that the physical metric is given solely in terms of the arbitrary addition to $\phi^{\mu\nu}$ as $\epsilon \to 0$. If $\bar{g}_{\mu\nu}$ is the metric of flat space-time in arbitrary coordinates, there exists at least one vector field ξ^μ such that

$$\bar{g}_{\mu\nu} = 2\xi_{(\mu;\nu)}.$$

Indeed, the necessary and sufficient integrability conditions

$$\bar{g}_{[\mu[\nu;\rho]\sigma]} = 0$$

are just the generalized SWG equations in a flat background, and are trivially satisfied. The non-zero physical solution is therefore not dictated by the curvature, but arises from an arbitrary choice of gauge. We conclude that Minkowski space-time is non-Machian.

Example 2. *Asymptotically flat space-time.* As we go to infinity, we have $R^{\lambda\mu}{}_{\rho\sigma} = O(\epsilon)$, and there exist vector fields ξ^μ such that $\underset{\xi}{\mathscr{L}} R^{\lambda\mu}{}_{\rho\sigma} = O(\epsilon)$. It is not possible to distinguish the addition of $2\nabla^{(\mu}\xi^{\nu)}$ to $\phi^{\mu\nu}$ from an SWG gauge transformation, so we expect $L^+[R] = O(\epsilon)$ and that the Minkowski part of the metric arises only if we arbitrarily make an addition to $L^+[R]$ of the allowed form. Indeed, if we choose Minkowski coordinates at infinity, we can use the weak field limit

$$g^{\mu\nu} \approx \eta^{\mu\nu} + \epsilon h^{\mu\nu}$$

$$R^{\mu\nu}{}_{\rho\sigma} \approx \epsilon \overset{(1)}{R}{}^{\mu\nu}{}_{\rho\sigma},$$

which is a good approximation in the asymptotic region. The generalized SWG equations then yield

$$\eta_{[\mu[\rho,\nu]\sigma]} = 0$$

$$h_{[\mu[\rho,\nu]\sigma]} = \overset{(1)}{R}_{\mu\nu\rho\sigma},$$

the second of which is the usual expression for the weak field Riemann tensor, while the first is the integrability condition for

$$\eta_{\mu\nu} = \xi_{\mu,\nu} + \xi_{\nu,\mu},$$

which, since it satisfies a homogeneous equation, is clearly not generated by physical sources. Modulo the usual difficulties of making precise statements about asymptotic behaviour in asymptotically flat space-times, it would seem to follow that such space-times are non-Machian.

Example 3. *Plane-wave space-times.* These space-times admit a vector field ξ^μ such that $\underset{\xi}{\mathscr{L}} R^{\lambda\mu}{}_{\rho\sigma} = 0$. It is known that the Riemann tensor of such a space-time does not uniquely determine a metric: $g_{\mu\nu}$ and $g_{\mu\nu} + 2\xi_{(\mu;\nu)}$ are two metrics with the same Riemann tensor (Hlavaty 1960; Collinson 1970). We therefore have the freedom to add $2\xi_{(\mu;\nu)}$ to $L^+[R]$ which implies the metrics are not generated solely by physical sources, and the space-times are non-Machian.

The results of these examples have no bearing on the local validity of the space-times concerned as solutions of the Einstein field equations, but they imply that as global, cosmological solutions such manifolds are incompatible with a natural interpretation of the Machian philosophy.

Most space-times will satisfy the first Mach condition. As an example we have: *Proposition* 4. The Robertson–Walker space-times satisfy the first Mach condition.

This result is obtained by noting that the symmetries imposed on the space-times lead to a unique functional form for a non-degenerate line-element up to coordinate transformations.† Indeed, we expect any metric derived in this way to satisfy the first Mach condition.

† Any ξ for which $\underset{\xi}{\mathscr{L}} R^{\lambda\mu}{}_{\rho\sigma} = 0$ is also a Killing vector and so does not change $g^{\mu\nu}$.

5. THE SECOND MACH CONDITION

The dynamical aspect of the problem consists in finding a Mach condition on the relation between the Riemann tensor and the stress tensor of a given cosmological model. The Ricci part of the Riemann tensor is given by the Einstein field equations with zero cosmological constant. While a cosmological term could be allowed formally, it would appear classically as an obviously non-Machian *source* term for the gravitational-inertial field. Such a term would represent a field which acted on everything, but was not in turn acted upon (Ellis 1971), and so would be an absolute element in the sense of Einstein (1954). For the Weyl part of the Riemann tensor, we have the identities:

$$C^{\lambda\mu\rho\sigma}{}_{;\sigma} = R^{\rho[\lambda;\mu]} - \tfrac{1}{6}g^{\rho[\lambda}R^{;\mu]} \tag{15}$$

equivalent to the Bianchi identities. The Weyl tensor is therefore a solution of the linear system

$$\psi^{\lambda\mu\rho\sigma}{}_{;\sigma} = \kappa(T^{\rho[\lambda;\mu]} - \tfrac{1}{3}g^{\rho[\lambda}T^{;\mu]}) \equiv \kappa J^{\lambda\mu\rho} \tag{16}$$

in the given curved background. In general it will be the only solution. If the system could be integrated by means of a Green function, an integral representation of the Weyl tensor would be obtained. A condition to be satisfied by a Machian space-time would then be that this representation should give the Weyl tensor as a linear functional of the currents, $J^{\lambda\mu\rho}$.

In Appendix 2 we show how this condition can be stated formally for a general globally-hyperbolic space-time. The condition is somewhat complicated and is incomplete in that it refers to a limiting process which is not well defined in general. The importance of the formal statement is to show that the Mach problem can be reduced to a well-defined problem on the nature of singularities in general relativity, and so is in principle rigorously soluble. For applications to simple cosmological models we shall use *ad hoc* methods coupled with the following informal statement of the second condition.

The second Mach condition (informal statement). A space-time will be said to satisfy the second Mach condition if the Weyl tensor is a linear functional of the matter currents $J^{\lambda\mu\rho}$ when regarded as a solution of the linear system (16) in the given space-time.

Proposition 5. All empty space-times are non-Machian.

Proof. In empty space-times we have $J^{\lambda\mu\rho} = 0$. Then the only possible candidate for a Machian solution to (16) is $\psi^{\lambda\mu\rho\sigma} = 0$, since only in this case is the field a linear functional of the sources. But this gives Minkowski space-time which fails to satisfy the first Mach condition.

If this were false our theory would be untenable. It is possible to have $J^{\lambda\mu\rho} = 0$ in non-empty space-times. Indeed, if the matter is a perfect fluid, these are just the Robertson–Walker solutions. The next theorem gives the only $J = 0$ Machian solutions.

Proposition 6. All conformally flat, non-flat, space-times satisfy the second Mach condition.

Proof. From the Bianchi identities

$$C^{\lambda\mu\rho\sigma}{}_{;\sigma} = \kappa J^{\lambda\mu\rho}$$

we see that the vanishing of the Weyl tensor implies the vanishing of the matter currents. The candidate for a Machian solution of

$$\psi^{\lambda\mu\rho\sigma}{}_{;\sigma} = 0$$

is $\psi^{\lambda\mu\rho\sigma} = 0$, which gives the stated result.

Combining this with Proposition 4, we obtain:

Corollary. All Robertson–Walker space-times are Machian.

In the next section we study the Bianchi cosmological models, which contain perfect fluids undergoing spatially homogeneous but anisotropic expansion. With a reasonable equation of state the matter content in most of these models is unimportant for the dynamical behaviour near the singular origin, so one would expect the solutions to be non-Machian. This turns out to be the case even for the exceptional models in which the matter dominates initially. A kinematical argument for expecting spatially homogeneous models with shear to be non-Machian is given by Bondi (1952).

Spatially homogeneous models with rotation are particularly important for the Mach programme. Here the matter rotates relative to the dynamical inertial frame provided by a Fermi-propagated tetrad. The homogeneity of the three-spaces implies that the kinematical inertial frame is precisely that which rotates with the matter at each point. Accordingly, such models are excluded by the Machian philosophy, and our task is to show that they are ruled out by our Mach conditions.

Finally, we ask whether there can be Machian solutions other than Robertson–Walker models; in particular, and of empirical importance, whether there are any inhomogeneous cosmological models. An investigation of the Bondi spherically symmetric solutions shows that indeed there are.

6. BIANCHI COSMOLOGIES

Our notation follows that of Ellis & MacCallum (1969). We assume a perfect fluid stress tensor

$$T_{\mu\nu} = (\mu+p)\, u_\mu u_\nu + p g_{\mu\nu} \tag{17}$$

and equation of state

$$p = (\gamma-1)\,\mu. \tag{18}$$

In general the rate of change of fluid velocity u_μ can be expressed in terms of the expansion θ, the shear $\sigma_{\mu\nu}$, the vorticity $\omega_{\mu\nu}$, and the acceleration α_μ, of the flow congruence as

$$\left.\begin{aligned} u_{\mu;\nu} &= \sigma_{\mu\nu} + \tfrac{1}{3}\theta(g_{\mu\nu}+u_\mu u_\nu) + \omega_{\mu\nu} - \alpha_\mu u_\nu \\ \sigma_{\mu\nu}u^\nu &= 0 = \omega_{\mu\nu}u^\nu; \quad \sigma_\mu{}^\mu = 0; \quad \sigma_{[\mu\nu]} = 0 = \omega_{(\mu\nu)} \end{aligned}\right\} \tag{19}$$

We introduce an orthonormal tetrad basis $\mathbf{e}_a$, with 'structure functions' $\gamma^a{}_{bc}$ defined by

$$[\mathbf{e}_a, \mathbf{e}_b] \equiv \underset{\mathbf{e}_a}{\mathscr{L}}\mathbf{e}_b = \gamma^c{}_{ab}\mathbf{e}_c.$$

The Jacobi identities are

$$[\mathbf{e}_a, [\mathbf{e}_b, \mathbf{e}_c]] + [\mathbf{e}_b, [\mathbf{e}_c, \mathbf{e}_a]] + [\mathbf{e}_c, [\mathbf{e}_a, \mathbf{e}_b]] = 0.$$

Spatial homogeneity is defined by the requirement that there exists a family of space-like hypersurfaces which are surfaces of transitivity of a group of motions. The principal tool is the choice of an orthonormal tetrad basis in which the geometrical quantities are functions of time only. The fluid flow vector $\mathbf{u} = \partial/\partial t$ is chosen as the time-like basis vector $\mathbf{e}_0$, and taken to be normal to the surfaces of homogeneity. This restricts the fluid congruence to have zero rotation and acceleration, which in turn implies

$$\gamma^0{}_{0i} = 0 = \gamma^0{}_{ij}.$$

For any choice of the triad $\mathbf{e}_i$ at a point of a surface of homogeneity $\mathscr{H}$, we obtain a basis at all points of $\mathscr{H}$ by dragging along under the action of the group. With this choice of tetrad, the Jacobi identities can be used to show that the structure functions depend on the cosmic time t only.

The γ_{i0j} are split into symmetric and antisymmetric parts

$$\gamma_{i0j} = -(\sigma_{ij} + \tfrac{1}{3}\delta_{ij}\theta) + \tau_{[ij]} \tag{20}$$

where $\Omega_i = \frac{1}{2}\epsilon_{ijk}\tau_{jk}$ is the angular velocity of the triad $\mathbf{e}_i$ with respect to a set of axes Fermi-propagated along $\mathbf{u}$.

The group types are classified by examining the functions $\gamma^i{}_{jk}$, which we write as

$$\gamma^i{}_{jk} = \epsilon_{jkl}n^{li} + \delta^i{}_k a_j - \delta^i{}_j a_k.$$

with n_{ij} symmetric. Class A models are defined to have $a_j = \gamma^i{}_{ji} = 0$, and Class B models have $a_j \neq 0$; type I solutions are characterized by $\gamma^i{}_{jk} = 0$. We do not require any further details of the classification.

The Jacobi identities applied to $(\mathbf{e}_0, \mathbf{e}_i, \mathbf{e}_j)$ yield evolution equations for the structure functions, which are, in a matrix notation

$$\left.\begin{aligned} \partial_0\mathbf{a} + \sigma\mathbf{a} + \tfrac{1}{3}\theta\mathbf{a} + \mathbf{a}\wedge\mathbf{\Omega} &= 0 \\ \partial_0 n + [n, \tau] - n\sigma - \sigma n + \tfrac{1}{3}\theta n &= 0, \end{aligned}\right\} \tag{21}$$

where the bracket notation is used for the matrix commutator. Applied to $(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k)$ the Jacobi identities yield

$$n\mathbf{a} = 0. \tag{22}$$

The Ricci tensor of the homogeneous three-spaces is found from the Gauss–Codacci equations to be

$$\tilde{R}_{ij} = -2\epsilon_{kl(i}n_{j)}{}^k a^l + 2n_{ik}n^k{}_j - n_{ij}(trn) - \delta_{ij}(2a_k a^k + n_{kl}n^{kl} - \tfrac{1}{2}(trn)^2). \tag{23}$$

The Einstein field equations determine the dynamics; the (0, 0) equation is

$$\dot{\theta} + \tfrac{1}{3}\theta^2 + tr(\sigma^2) + \tfrac{1}{2}\mu + \tfrac{3}{2}p = 0, \tag{24}$$

the trace-free part of the (i, j) equations are

$$\partial_0\sigma + \theta\sigma + [\sigma, \tau] + \tilde{R} - \tfrac{1}{3}(tr\tilde{R})\,\mathbb{1} = 0, \tag{25}$$

and the trace,

$$\tfrac{1}{3}\theta^2 = \tfrac{1}{2}tr(\sigma^2) - \tfrac{1}{2}tr(\tilde{R}) + \mu \tag{26}$$

is a first integral of (24). Finally, the $(0, i)$ equations are

$$3\sigma\mathbf{a} - \text{vec}\,[\sigma, n] = 0, \tag{27}$$

where, for any matrix β,

$$(\text{vec}\,\beta)_i = \tfrac{1}{2}\,\epsilon_{ijk}\beta_{jk}. \tag{28}$$

To illustrate the line of argument we consider first the simplest case of the type I solutions with metric

$$ds^2 = -dt^2 + X(t)\,dx^2 + Y(t)\,dy^2 + Z(t)\,dz^2.$$

Theorem 2. Bianchi Type I cosmologies are non-Machian.

Proof. In type I we have $n = 0$, and we can choose a frame with σ diagonal. The evolution equations for n (21), show that Ω is arbitrary, so can be chosen to be zero. A length scale, l, is defined by

$$\theta = \frac{3\dot{l}}{l}, \quad \dot{l} \equiv \frac{dl}{dt},$$

and used as the independent variable.

The field equations (25) can be integrated to give the evolution of the shear, $\sigma = \text{diag}(\sigma_i)$,

$$\sigma_i = l^{-3}\Sigma_i, \quad \Sigma_i = \text{constant}.$$

The matter conservation equation, which for a perfect fluid reads

$$\dot{\mu} + (\mu + p)\,\theta = 0 \tag{29}$$

integrates to

$$\mu = \mu_0 l^{-3\gamma}, \quad \mu_0 = \text{constant},$$

and (26) then defines l as a function of cosmic time.

We now compute the field equations (16) in these models. The equations

$$e_j^{\mu}\nabla_{\mu}\psi^{0i0j} = \kappa J^{0i0}, \quad e_k^{\mu}\nabla_{\mu}\psi^{ij0k} = \kappa J^{ij0} \tag{30}$$

reduce to algebraic constraints which are satisfied by taking

$$\Psi_{ij} \equiv \psi_{0i0j} - \tfrac{1}{2}i\epsilon_{klj}\psi_{0i}{}^{kl}$$

to be diagonal. The remaining field equations determine the evolution of $(\Psi_{ij}) = \text{diag}(\Psi_i)$:

$$\partial_0\begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\end{pmatrix} + \begin{pmatrix}\frac{2}{3}\theta - \sigma_1 & -\sigma_3 - \frac{1}{3}\theta & -\sigma_2 - \frac{1}{3}\theta\\ -\sigma_3 - \frac{1}{3}\theta & \frac{2}{3}\theta - \sigma_2 & -\sigma_1 - \frac{1}{3}\theta\\ -\sigma_2 - \frac{1}{3}\theta & -\sigma_1 - \frac{1}{3}\theta & \frac{2}{3}\theta - \sigma_3\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\end{pmatrix} = \tfrac{1}{2}(\mu + p)\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix}. \tag{31}$$

The known Weyl tensor is

$$\begin{pmatrix}V_1\\ V_2\\ V_3\end{pmatrix} = \begin{pmatrix}-\frac{1}{3}\theta\sigma_1 + \sigma_1{}^2 + \frac{1}{3}\sigma_i\sigma^i\\ -\frac{1}{3}\theta\sigma_2 + \sigma_2{}^2 + \frac{1}{3}\sigma_i\sigma^i\\ -\frac{1}{3}\theta\sigma_3 + \sigma_3{}^2 + \frac{1}{3}\sigma_i\sigma^i\end{pmatrix}.$$

Expanding this in inverse powers of l, one obtains

$$V_i = l^{-6}\{-\tfrac{1}{3}(\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2) + \Sigma_i^2 - 1/\sqrt{6}\,\Sigma_i(\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2)^{1/2}\} + l^{-3\gamma}\{\ldots\} + \ldots$$

It is straightforward to check that the term in l^{-6} is a homogeneous solution of the evolution equations (31) to this order. If the shear is non-zero, this term cannot vanish, so the Weyl tensor contains a part which is not a linear functional of the matter currents, and the models are non-Machian.

Corollary. There exist non-Machian non-empty solutions with closed spatial sections.

Proof. The homogeneous three-spaces in Bianchi type I models are flat ($R = 0$) and can be given the topology of a torus by suitable identifications.

As a result of theorem 3 it will follow that there exist non-empty non-Machian solutions with closed three-spaces without identifications of points (Type IX Bianchi models).

The method of proof of Theorem 2 requires a knowledge of the asymptotic form of the solution of the Einstein field equations. In general this is not available and a proof which depends only on the structure of the equations is required.

Theorem 3. The perfect fluid non-rotating Bianchi models are non-Machian.

Proof. The field equations (16) for ψ^{abcd} yield a set of algebraic constraints (30), which can be solved for the off-diagonal elements of $\Psi_{ij} = \psi_{0i0j} - \frac{1}{2}i\epsilon_{klj}\psi_{0i}{}^{kl}$, and a set of evolution equations. For two matrices A, B, we define

$$[A \wedge \Sigma]_{ij} = \epsilon_{ilm}\epsilon_{jkm}A_{lk}B_{mn}.$$

The evolution equations can then be written

$$-\partial_0\Psi + [\tau + ia, \Psi] + [(\sigma - \tfrac{1}{2}in) \wedge \Psi] + \tfrac{1}{2}(\sigma - in)\Psi + \tfrac{1}{2}\Psi(\sigma - in) - \theta\Psi = -\tfrac{1}{2}(\mu + p)\sigma \quad (32)$$

where vec(a) = a (equation (28)). The known Weyl tensor is

$$V = -\tfrac{1}{3}\theta\sigma - \tilde{R} + \tfrac{1}{3}(tr\tilde{R} - tr(\sigma^2))\,\mathbb{1} + \sigma^2 + i\,\{tr(\sigma n)\,\mathbb{1} + \tfrac{1}{2}(trn)\,\sigma - \tfrac{3}{2}(\sigma n + n\sigma) - \tfrac{1}{2}[\sigma, a]\}.$$

This is not explicitly a function of time, but depends on time only through μ, θ, $\mathbf{\Omega}$, $\mathbf{a}$, σ and n. If we choose a frame with $n = \mathrm{diag}(n_i)$, then θ and $\mathbf{\Omega}$ can be solved for algebraically from the (0, 0) field equation (26) and the Jacobi identities (21). The off-diagonal components of σ are determined algebraically by the (0, i) field equations (27). The Weyl tensor can therefore be regarded as a function of μ, $\mathbf{a}$, σ_i and n_i, with the time dependence of these quantities given by the conservation equation (29), and the remaining Jacobi identities and field equations.

We now make the scale transformation

$$\mu \to \lambda\tilde{\mu}.$$

Regarded as functions of an independent time variable our quantities will be complicated functions of λ. However, we can write

$$\partial_0\Psi = (\partial_0\mu)\,\partial_\mu\Psi + \sum_i [(\partial_{0\sigma i}\,\partial\sigma_i)\,\Psi + (\partial_{0n_i}\,\partial_{ni})\,\Psi + (\partial_0 a_i\,\partial_{a_i})\,\Psi],$$

and substitute for the time derivatives to obtain from (32) a partial differential system with coefficients analytic in λ. Since the known Weyl tensor is analytic in λ, as a function of $\tilde{\mu}$, σ, n, $\mathbf{a}$, we expand V as a power series in λ:

$$V = \sum_{p=0\,(p)}^{\infty} V(\tilde{\mu}, \sigma\ n, \mathbf{a})\,\lambda^p.$$

It is straightforward to see that

$$\underset{(0)}{V} = -1/\sqrt{6}\,(tr(\sigma^2)-tr\tilde{R})^{1/2}\,\sigma-\tilde{R}+\sigma^2+\tfrac{1}{3}(tr\tilde{R}-tr(\sigma^2))\,\mathbb{1}$$
$$+i\{tr(\sigma n)\,\mathbb{1}+\tfrac{1}{2}tr(n)\,\sigma-\tfrac{3}{2}(\sigma n+n\sigma)-\tfrac{1}{2}[\sigma, a]\}$$

is a solution of the homogeneous evolution equations to zero order in λ.

Finally we have to show that $\underset{(0)}{V}$ cannot be identically zero. To do this, observe that $\underset{(0)}{V} \equiv 0$ is invariant under the scaling $\sigma \to \alpha\sigma$, $n \to \alpha n$, $a \to \alpha a$, but that the evolution equations are not so invariant if the shear is non-zero. It follows that $\underset{(0)}{V} \equiv 0$ would imply a functional relationship incompatible with the evolution of the system. This completes the proof.

7. SPATIALLY HOMOGENEOUS ROTATING MODELS

If the matter content of a Bianchi model is allowed to possess vorticity there are two natural time-like congruences associated with the normals to the surfaces of homogeneity and with the fluid flow. We choose an orthonormal tetrad associated with the normal congruence as in Section 6, and use Σ_{ij}, Θ to denote the shear and expansion. The shear, expansion and vorticity of the fluid congruence are denoted by σ_{ab}, θ and ω_{ab}. By substituting for the structure functions in the definition of these quantities (19), one finds a relation between the two sets of variables involving the fluid velocity vector $(u^a) = (u^0, \mathbf{u})$.

The Jacobi identities (21) are unchanged except that they must be written to refer to the normal congruence. The Einstein equations are modified by additional source terms; in particular the $(0, i)$ equations become

$$\text{vec}\,[n, \Sigma]+3\Sigma\mathbf{a} = (\mu+p)\,u^0\mathbf{u}. \qquad (33)$$

There is an additional conservation equation

$$(\mu+p)\,u^\mu\nabla_\mu\mathbf{u}+(u^\mu\nabla_\mu p)\,\mathbf{u} = 0. \qquad (34)$$

We choose a frame in which $\mathbf{u} = (v, 0, 0)$ and $n_{23} = 0$. In flat space-time this would be a frame which rotated with matter. This choice simplifies the (non-linear) conservation equations at the expense of complicating the (linear) Jacobi identities. The conservation equations (34) now give Ω_2, Ω_3 algebraically in terms of v, Σ, n, $\mathbf{a}$, μ, Θ. The (2, 3) Jacobi identities (21) give Ω_1. The $(0, i)$ field equations yield the off-diagonal components of Σ in terms of those of n. Substituting in the (1, 2), (2, 3), (3, 1) field equations and eliminating the derivatives of n_{12}, n_{13} through the Jacobi identities gives algebraic equations for n_{12}, n_{13}. Θ is given algebraically by the first integral of the (0, 0) field equation (26). Thus v, μ and $\mathbf{a}$, and the diagonal components of Σ and n are subject to the conditions $\Sigma_{11}+\Sigma_{22}+\Sigma_{33} = 0$ and $n\mathbf{a} = 0$, and all the remaining variables are expressed algebraically in terms of these basic variables.

We can now proceed as in Theorem 3: the constraint equations (30) can be solved for the off-diagonal components of Ψ, and the evolution equations (32) can be regarded as partial differential equations for $\Psi(\Sigma_i, n_i, \mathbf{a}, v, \mu)$, the known Weyl tensor, V, being formally unchanged by the introduction of vorticity when expressed in terms of the normal congruence. We introduce again the scaling $\mu \to \lambda\tilde{\mu}$ to obtain a system of differential equations analytic in λ, and expand V as a power series in λ. The term independent of λ, $\underset{(0)}{V}$, is a solution of the homogeneous system to zero order in λ.

Finally, the (0, i) field equations (33) show that non-zero vorticity implies $\Sigma \neq 0$, so, as in Theorem 3, $\underset{(0)}{V}$ cannot be identically zero. We have therefore outlined a proof of:

Theorem 4. The perfect fluid Bianchi cosmologies with vorticity are non-Machian.

This result states not only that the vorticity vanishes on average in a spatially homogeneous Machian solution, but that zero spatial gradients at a point require zero vorticity there. This means we cannot compensate for non-zero rotation within a particle horizon by the motion of matter outside the horizon. It is not unreasonable to assume that small departures from homogeneity in a Machian universe would imply small vorticities. The observed isotropy of the microwave background shows that at no time in its history can the Universe have been rotating on a time scale shorter than the expansion time scale (Collins & Hawking 1973a; Sciama 1973). The precise number is model dependent, but it appears that the small vorticity cannot be accounted for by dissipative processes. Since the absence of significant vorticity would seem to be explained in terms of Mach's Principle, this provides us with the strongest observational evidence in favour of the Principle.

8. BONDI MODELS

Finally, we consider Bondi spherically symmetric models (Bondi 1947) as the simplest examples of inhomogeneous cosmologies. While they are not realistic models of the actual Universe, they show interesting features which may be expected to be present in more complex situations.

We follow the discussion of Eardley, Liang & Sachs (1972). The matter content is taken to be a pressure-free perfect fluid with density $\mu(r, t)$. The general spherically symmetric metric is written

$$ds^2 = -dt^2 + A^2(r, t)\, d\Omega + B^2(r, t)\, dr^2; \quad d\Omega = d\theta^2 + \sin^2\theta\, d\phi^2.$$

From the field equations we obtain

$$A' = [1 + \beta(r)]\, B, \quad \beta(r) > -1$$

$$2A\ddot{A} + \dot{A}^2 - \beta = 0,$$

where A' denotes $\partial A/\partial r$ and $\dot{A}$ denotes $\partial A/\partial t$. The second equation is integrated to give

$$A(\dot{A}^2 - \beta) = \lambda(r),$$

where, from the Bianchi identities,

$$\lambda' = \mu_0(1+\beta)^{1/2}$$

with

$$\mu_0 = A^2 B\mu. \tag{35}$$

The functions $\beta(r)$ (or $\lambda(r)$) and $\mu(r)$ are the arbitrary initial data.

From considerations of symmetry we expect the first Mach condition to be satisfied and we assume this to be the case. We expect no radiation field owing to the spherical symmetry, so it is the constraint equations (30) that are relevant to a discussion of the second Mach condition.

Eardley *et al.* (1972) show how to assign a positive definite three-metric to the 'initial' singularity present in these models by expanding the space-time metric in terms of a cosmic time, $t - {}_0t(r)$, relative to a shifted origin $t = {}_0t(r)$ on each world-line, and taking the coefficient of the leading term. Three types of singularity are found to occur, labelled as Heckmann–Schücking, Robertson–Walker, or Kasner, according to the exact solution which coincides with the first approximation for a suitable choice of certain integration functions. In order to satisfy the second Mach condition it is necessary that the leading term in the known Weyl tensor should solve to this order the Bianchi identities with the dominant term in the matter density as source function. Further terms are obtained by iteration with no new parameters. It follows that the Machian behaviour of the leading term is also a sufficient Mach condition.

Theorem 5. Pressure free, perfect fluid Bondi models with Robertson–Walker type singularities are Machian; those with Heckmann–Schücking or Kasner types are non-Machian.

Proof. We define an orthonormal tetrad $\mathbf{e}_a^\mu$ by

$$\mathbf{e}_0 = \frac{\partial}{\partial t}, \quad \mathbf{e}_1 = B^{-1}\frac{\partial}{\partial r}, \quad \mathbf{e}_2 = A^{-1}\frac{\partial}{\partial \theta}, \quad \mathbf{e}_3 = (A \sin\theta)^{-1}\frac{\partial}{\partial \phi},$$

with respect to which the field equations (30) have the explicit form

$$B^{-1}(\Psi_{11})' + 2A'A^{-1}B^{-1}\Psi_{11} + A^{-1}\cot\theta\,\Psi_{12} - A'A^{-1}B^{-1}(\Psi_{22}+\Psi_{33}) = 3B^{-1}\mu'$$
$$B^{-1}(\Psi_{12})' + 3A'A^{-1}B^{-1}\Psi_{12} + i(B^{-1}\dot{B} - A^{-1}\dot{A})\,\Psi_{13} + A^{-1}\cot\theta(\Psi_{22}-\Psi_{33}) = 0$$
$$B^{-1}(\Psi_{13})' - i(B^{-1}\dot{B} - A^{-1}\dot{A})\,\Psi_{12} + 3A'A^{-1}B^{-1}\Psi_{13} + 2A^{-1}\cot\theta\,\Psi_{23} = 0.$$

Define

$$\Psi_{\pm 2} = \Psi_{22} \pm 2i\Psi_{23} - \Psi_{33}$$
$$\Psi_{\pm 1} = \Psi_{12} \pm i\Psi_{13}$$
$$\Psi_0 = \Psi_{11}.$$

Using $\Psi_{11}+\Psi_{22}+\Psi_{33} = 0$, the spin-0 constraint is

$$B^{-1}(\Psi_0)' + 3A'A^{-1}B^{-1}\Psi_0 + \tfrac{1}{2}A^{-1}\cot\theta(\Psi_{+1}+\Psi_{-1}) = \tfrac{1}{3}B^{-1}\mu'. \tag{36}$$

We take $\Psi_{\pm 2} = 0$ since the spherical symmetry implies the free dynamical modes of the Weyl tensor must vanish. The Machian solution of the spin-1 constraint is then $\Psi_{\pm 1} = 0$ so (36) becomes

$$(A^3\Psi_0)' = \tfrac{1}{3}A^3\mu' \tag{37}$$

with μ given by (35).

From the Ricci identities applied to $\mathbf{e}_0$, we calculate the Weyl tensor:

$$(E_{ij}) = \mathrm{diag}(-\tfrac{2}{3}(B^{-1}\ddot{B} - A^{-1}\ddot{A}), \quad -\tfrac{1}{3}(A^{-1}\ddot{A} - B^{-1}\ddot{B}), \quad -\tfrac{1}{3}(A^{-1}\ddot{A} - B^{-1}\ddot{B}))$$
$$(H_{ij}) = 0. \tag{38}$$

We investigate in detail the case $\beta = 0$, the other cases ($\beta > 0$; $-1 < \beta < 0$) being analogous.

(i) ${}_0t'(r) = 0$ (Robertson–Walker type singularity): writing τ for $t - {}_0t(r)$, we have

$$A = \bar{A}(r)\,\tau^{2/3}, \quad B = \bar{B}(r)\,\tau^{2/3}.$$

Suppose $\Psi_0 = \bar{\Psi}_0(r)\,\tau^{\alpha}$; then the left-hand side of (37) is

$$(A^3\Psi_0)' = (\bar{A}^2\bar{\Psi}_0)'\,\tau^{\alpha+2},$$

and the right-hand side becomes

$$A^3\mu' = (\mu_0 A^{-2}B^{-1})'.$$

It follows that the Machian solution is $\alpha = -2$, and this is just the dependence found for the known Weyl tensor (38).

(ii) ${}_0t'(r) \neq 0$ (Heckmann–Schücking singularity): we have

$$A = \bar{A}(r)\,\tau^{2/3}, \quad B = \bar{B}(r)\,\tau^{-1/3}$$

from which

$$A^3\mu' = \bar{A}^3(\mu_0\bar{A}^{-1}\bar{B}^{-1})'\,\tau - \bar{A}^2\mu_0\bar{B}^{-1}{}_0t'.$$

If $\Psi_0 = \bar{\Psi}_0(r)\,\tau^{\alpha}$, then

$$(A^3\Psi_0)' = \bar{\Psi}_0'\tau^{\alpha+2} + {}_0t'\tau^{\alpha+1}(\alpha+2)\,\bar{\Psi}_0.$$

A Machian solution would require $\alpha = -1$, but for the known Weyl tensor we calculate $\alpha = -2$. Hence the leading term satisfies the homogeneous equation to this order and the model is non-Machian.

This example shows how a central condensation may be detected by the ' Coulomb ' part of the Weyl tensor, even though the major part of the condensation may be beyond the particle horizon of the observer. It is difficult to see how any such solutions could satisfy Gilman's Mach condition (Section 1).

9. CONCLUSION

The preceding results show that Mach's Principle rules out anisotropic expansion (Theorem 3) or rotation (Theorem 4) in spatially homogeneous perfect fluid cosmologies, and restricts the type of singularity in a simple class of inhomogeneous models (Theorem 5). This leads one to propose that Mach's Principle, fully incorporated into general relativity, requires our model of the Universe to be a Robertson–Walker solution with local irregularities. This would provide an alternative to ' chaotic cosmology ' (Misner 1968) or the ' Dicke–Carter philosophy ' (Collins & Hawking 1973b) as an explanation of the observed isotropy of the Universe, and, if true, would take us a considerable way towards an understanding of its large-scale structure.

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REFERENCES

Al'tshuler, B. L., 1967. *Sov. Phys., J.E.T.P.*, **24**, 766.
Bondi, H., 1947. *Mon. Not. R. astr. Soc.*, **107**, 410.
Bondi, H., 1952. *Cosmology*, University Press, Cambridge.
Choquet-Bruhat, Y., 1968. *Battelle Rencontres*, Benjamin, New York.
Collins, C. B. & Hawking, S. W., 1973a. *Mon. Not. R. astr. Soc.*, **162**, 307.
Collins, C. B. & Hawking, S. W., 1973b. *Astrophys. J.*, **180**, 317.
Collinson, C. D., 1970. *J. math. Phys.*, **11**, 818.
Deser, S., 1967. *Ann. Inst. Henri Poincare* **VII**, 149.
Desoer, V. C. A. & Whalen, B. H., 1963. *J. SIAM*, **11**, 442.
DeWitt, B. S. & Brehme, R. W., 1960. *Ann. Phys.*, **9**, 220.
Eardley, D., Liang, E. & Sachs, R. K., 1972. *J. math. Phys.*, **13**, 99.
Einstein, A., 1954. Foreword to *Concepts of space*, M. Jammer, Harvard University Press.
Ellis, G. F. R., 1971. *General relativity and cosmology*, Proceedings of the International Summer School of Physics, 'Enrico Fermi', Course XLVII, Academic Press, London and New York.
Ellis, G. F. R. & MacCallum, M. A. H., 1969. *Comm. Math. Phys.*, **12**, 108.
Geroch, R., 1970. *J. math. Phys.*, **11**, 437.
Gilman, R. C., 1970. *Phys. Rev.*, **D2**, 1400.
Hawking, S. W. & Ellis, G. F. R., 1973. *The large scale structure of space-time*, University Press, Cambridge.
Hlavaty, V., 1960. *J. Math. Mech.*, **9**, 89.
Lynden-Bell, D., 1967. *Mon. Not. R. astr. Soc.*, **135**, 453.
Misner, C. W., 1968. *Battelle Recontres*, Benjamin, New York.
Penrose, R., 1954. *Proc. Cam. Phil. Soc.*, **51**, 406.
Penrose, R., 1964. *Relativity, groups and topology*, the 1963 Les Houches Lectures, Gordon and Breach, New York.
Petryshyn, W. V., 1967. *J. Math. Anal.*, **18**, 417.
Sciama, D. W., Waylen, P. C. & Gilman, R. C., 1969. *Phys. Rev.*, **187**, 1762.
Sciama, D. W., 1973. *Colloques Internationaux C.N.R.S. No.* **220**, — Ondes et radiations gravitationnelles, 267.

APPENDIX 1

THE EXISTENCE OF THE GENERALIZED INVERSE

Relative to an arbitrarily chosen positive definite metric on M, select an orthonormal vector basis E_j, and hence a basis for tensor fields:

$$E^I{}_J = E^{i_1} \otimes \ldots \otimes E^{i_I} \otimes E_{j_1} \otimes \ldots \otimes E_{j_J}.$$

Given a tensor field W, with components $W^I{}_J = W(E^I{}_J)$, define

$$|W| = |W^I{}_J E_I{}^J|.$$

If Ω is an open set of M, with compact closure, and

$$\sum_{|K|=k} D^K W$$

is the sum of partial derivatives of W of order k, then

$$\|W\|^q{}_p = \left\{\int_\Omega \sum_{|K|\leqslant p} |D^K W|^q \, d\sigma\right\}^{1/q},$$

where $d\sigma$ is the volume element on Ω induced by the chosen metric, is a norm if the integral exists. The Sobolev spaces $W^q{}_p(\Omega)$ are the completions of spaces of tensor fields of type (r, s), for fixed r and s, having qth power integrable derivatives of order p over Ω. The Banach structure of these spaces is independent of the choice of positive definite metric and basis vector fields (Hawking & Ellis 1973), but the Hilbert space structure of $H_p(\Omega) \equiv W^2{}_p(\Omega)$ is not. The choice of Hilbert structure is equivalent to a choice of gauge. We shall denote by H_p the Hilbert spaces of tensors of type (2, 2).

We symbolize the generalized SWG equations (4, 5) as

$$\mathscr{L}(Z) = K, \tag{A1.1}$$

and the consistency conditions (the varied Bianchi identities) as $\bar{D}K = 0$. In the following we shall understand K to stand for an equivalence class of sources, two sources belonging to the same class if they are related by a gauge transformation (6). This enables us to write $ker(\mathscr{L})$ for the set of gauge transformations of the superpotential Z.

In the Bianchi gauge, $DZ = 0$ (Proposition 1), we obtain the hyperbolic system (9), which we write as

$$L(Z) = K. \tag{A1.2}$$

From the general theory of hyperbolic equations it is known that with certain restrictions on the domain Ω, on the coefficients and initial conditions, and for sufficiently large p (>4), this equation has a solution $Z \in H_p$ for any $K \in H_{p-1}$ (see Hawking & Ellis (1973) Proposition 7.4.7). We restate this as:

Proposition A1. L maps a dense subset of H_p on to H_{p-1}.

The usual method of obtaining a solution of (A1.1) is essentially to show that the gauge conditions $DZ = 0$ can be satisfied by the addition of an element of *ker*(L) to a solution of (A1.2), with K restricted by $\bar{D}K = 0$. Lemma A1 is a restatement of this. The essence of the generalized inverse is to achieve this instead by a projection (and hence linear) map.

Lemma A1. The range space of $\mathscr{L} : H_p \to H_{p-1}$, is the kernel of $\bar{D} : H_{p-1} \to W^1_{p-2}$.

Proof. If $K \in H_{p-1}$ and $K \in \mathscr{R}(\mathscr{L})$ then $\bar{D}K = 0$ so $\mathscr{R}(\mathscr{L}) \subset ker\bar{D}$. Conversely, if $K \in H_{p-1}$ then there exists a $Z \in H_p$ satisfying (A1.2) by Proposition A1. If we *restrict* K by $\bar{D}K = 0$ and restrict the *initial* conditions by $DZ = 0$, then we obtain a solution Z' (say). Propositions 2 and 3 imply that Z' satisfies (A1.1) and Proposition 1 ensures that $Z' \in ker\bar{D}$.

Lemma A2. The kernel of $\bar{D} : H_{p-1} \to W^1_{p-2}$ is closed.

Proof. $Ker(\bar{D})$ is the inverse image of $0 \in W^1_{p-2}$ by a continuous map.

Similarly, since $\mathscr{L} : H_p \to W^1_{p-2}$ is continuous, $ker(\mathscr{L})$ is closed.

Thus we can write $H_p = (ker(\mathscr{L})) \oplus (ker(\mathscr{L}))^\perp$, where $\perp$ denotes orthogonal complement, and $H_{p-1} = (ker(\bar{D})) \oplus (ker(\bar{D}))^\perp = \mathscr{R}(\mathscr{L}) \oplus (\mathscr{R}(\mathscr{L}))^\perp$. Therefore, we have the analogue of proposition A1:

Proposition A2. $\mathscr{L}$ maps a dense subset of $(ker(\mathscr{L}))^\perp$ in H_p to $ker\bar{D}$ in H_{p-1}.

It is now possible to construct a generalized inverse as in Desoer & Whalen (1963), the only difference being that the domain of our mapping is a dense subset of $(ker(\mathscr{L}))^\perp$ rather than the whole of it.

If $K \in ker(D)$ in H_{p-1} then there exists a Z in $(ker(\mathscr{L}))^\perp$ such that $\mathscr{L}(Z) = K$, and we put $\mathscr{L}^+(K) = Z$. It is easy to see that Z is unique; if $K \in (ker(D))^\perp$, then we define $\mathscr{L}^+(K)$ to be $0 (\in H_p)$. $\mathscr{L}^+$ is a generalized inverse in the sense of Moore and Penrose (see Desoer & Whalen 1963). This proves Theorem 1. The inverse obtained depends on the Hilbert space structure of H_p; but a second inverse can differ from $\mathscr{L}^+(K)$ by an element of $(ker(\mathscr{L}))^\perp$ only, which is a gauge transformation.

APPENDIX 2

THE SECOND MACH CONDITION

By differentiating equation (16) and re-arranging terms we obtain

$$\Box \psi_{\lambda\mu\rho\sigma}+2R^{\alpha}{}_{[\lambda}{}^{\beta}{}_{[\rho}\psi_{\mu]\sigma]\alpha\beta}+R^{\alpha}{}_{[\lambda}\psi_{\mu]\alpha\rho\sigma}+R^{\alpha}{}_{[\rho}\psi_{\sigma]\alpha\lambda\mu}$$
$$= 2\kappa\{J_{[\lambda\mu][\rho;\sigma]}+\nabla^{\nu}J_{\nu[\sigma[\mu}g_{\rho]\lambda]}+J_{[\rho\sigma][\lambda;\mu]}+\nabla^{\nu}J_{\nu[\mu[\sigma}g_{\lambda]\rho]}\} \equiv \kappa J_{\lambda\mu\rho\sigma}.$$

The initial value problem for this system is solved locally by means of a retarded Green function $G^{\alpha'\beta'\gamma'\delta'}_{\mu\nu\rho\sigma}$ having the algebraic symmetries of the Riemann tensor to give (DeWitt & Brehme 1960).

$$\psi^{\alpha'\beta'\gamma'\delta'}(x') = \kappa\int_V G^{\alpha'\beta'\gamma'\delta'}_{\mu\nu\rho\sigma}\, J^{\mu\nu\rho\sigma}\sqrt{-g}\, d^4x+\int_s (\psi^{\mu\nu\rho\sigma}\nabla_{\lambda}G^{\alpha'\beta'\gamma'\delta'}_{\mu\nu\rho\sigma}$$
$$-G^{\alpha'\beta'\gamma'\delta'}_{\mu\nu\rho\sigma}\,\nabla_{\lambda}\psi^{\mu\nu\rho\sigma})\sqrt{-g}\, dS^{\lambda} \quad \text{(A2.1)}$$

Equations (16) now represent constraints on the data on the initial surface S. In general the constraints will not be consistent with the evolution equations unless $\psi_{\lambda\mu\rho\sigma}$ is the Weyl tensor, so that in general the constraints will not be preserved. Of course, we are interested in this representation only when $\psi_{\lambda\mu\rho\sigma}$ *is* the Weyl tensor.

We introduce a gaussian coordinate system based on S which then has intrinsic metric $h_{ij} = g_{ij}$ and the second fundamental form $K_{ij} = \frac{1}{2}(\partial g_{ij}/\partial t)$. With the definition $\Psi_{ij} = \psi_{0i0j}-\frac{1}{2}i\eta_{klj}\psi_{i0}{}^{kl}$ the constraints which do not involve derivatives out of S can be written

$$\Psi_i{}^j{}_{|j}+\tfrac{1}{2}i\eta_{ikl}K^l{}_j\Psi^{jk} = \kappa(J_{0i0}+i\,{}_*J_{0i0}) \quad \text{(A2.2)}$$

where the vertical bar denotes covariant differentiation in the t = constant three-spaces. We want to decompose $\Psi_i{}^j$ covariantly with respect to coordinate transformations in S into constraint variables to be eliminated by solving (A2.2) and dynamical variables which will be freely specifiable data. The following method leads to a self-adjoint form for the constraints (A2.2), involving only the constraint variables.

We introduce the notation $\Lambda^i{}_{kj} = \frac{1}{2}K_{l(j}\eta_{k)}{}^{li}$, and define a ' covariant derivative ', $\mathscr{D}_k$, by means of the (symmetric) ' connection ' $\Gamma^i{}_{jk}+\Lambda^i{}_{jk}$: for a scalar field ϕ, $\mathscr{D}_k\phi \equiv \partial_k\phi$ and for a contravariant vector field V^i,

$$\mathscr{D}_kV^i = \partial_kV^i+(\Gamma^i{}_{kj}+\Lambda^i{}_{kj})\, V^j,$$

with the usual extension to general tensor fields, using $\mathscr{D}_k(V^iU_i) = \partial_k(V^iU_i)$. Since $\mathscr{D}_kh_{ij} \neq 0$ care must be exercised in raising and lowering indices. However, we do have $\mathscr{D}_jh^{ij} = 0$, so we can define $\mathscr{D}^j \equiv h^{jk}\mathscr{D}_k$ and write $\mathscr{D}_jV^j = \mathscr{D}^jV_j$. The constraints (A2.2) now read

$$\mathscr{D}^j\Psi^i{}_j = \kappa J^i \quad \text{(A2.3)}$$

which suggests the decomposition (compare Deser 1967):

$$\Psi^i{}_j = \overset{\mathrm{T}}{\Psi}{}^i{}_j+\mathscr{D}^i\xi_j+\mathscr{D}_j\xi^i-\tfrac{2}{3}\delta^i{}_j\mathscr{D}_m\xi^m$$

$$\mathscr{D}^j\overset{\mathrm{T}}{\Psi}{}^i{}_j \equiv 0 \equiv \overset{\mathrm{T}}{\Psi}{}^i{}_i.$$

Substituting in (A2.3) we obtain an elliptic system for the constraint variables ξ^j:

$$(\mathscr{D}_k\mathscr{D}^k\delta^i{}_j+\tfrac{1}{3}\mathscr{D}^i\mathscr{D}_j)\,\xi^j+\tilde{R}^i{}_k\xi^k = \kappa J^i \qquad \text{(A2.4)}$$

where $\tilde{R}_{ij}$ is the Ricci contraction of the 'curvature tensor' $\tilde{R}^k{}_{ilj}$ formed from the 'connection' $\Gamma^i{}_{jk}+\Lambda^i{}_{jk}$. Explicit calculation shows that $\tilde{R}_{ij}$ is symmetric, so the system is formally self-adjoint.*

By means of a (complex) vector Green function $H^{k'}_i(x', x)$, were present the solution of (A2.4) as

$$\xi^{k'}(x') = \kappa\int_s H^{k'}_i J^i\sqrt{h}\,d^3x+\int_{\partial s}(\xi^i\mathscr{D}_kH^{k'}_i+\tfrac{1}{3}\xi_k\mathscr{D}^iH^{k'}_i-H^{k'}_i\mathscr{D}_k\xi^i$$
$$-\tfrac{1}{3}H^{k'}_k\mathscr{D}_i\xi^i)\sqrt{h}\,dS^k. \qquad \text{(A2.5)}$$

In a globally hyperbolic space-time (Geroch 1970) having S as a global Cauchy surface, a necessary Mach condition will be that the integral over ∂S should vanish. This cannot be applied directly if S is not closed, since the boundary values for ξ^i or its derivatives are unknown. Instead, we aim to formulate the condition that the surface term in (A2.1) should be a linear function of the matter currents. Thus we put

$$\overset{\mathrm{L}}{\Psi}{}^i{}_j = \mathscr{D}^i\xi_j+\mathscr{D}^j\xi_i-\tfrac{2}{3}\delta^i{}_j\mathscr{D}_k\xi^k \qquad \text{(A2.6)}$$

with ξ^i now *defined* by the inhomogeneous term in (A2.5):

$$\xi^i = \int_s H^i{}_{k'}J^{k'}\sqrt{h}\,d^3x'. \qquad \text{(A2.7)}$$

With the notation

$$U^{\Gamma'}_{ij} = G^{\alpha'\beta'\gamma'\delta'}_{0i0j}+\tfrac{1}{2}ih_{kj}\eta^{0k\lambda\mu}G^{\alpha'\beta'\gamma'\delta'}_{0i\lambda\mu}$$

and using (16), we can write the integrand of the surface integral in (A2.1) as

$$8\mathscr{R}e\{\Psi^{ij}(\partial_0U^{\Gamma'}_{ij}-2K^l{}_iU^{\Gamma'}_{lj})+iU^{\Gamma'}_{ij}\eta^{jkl}\mathscr{D}_k\Psi^i{}_l+\kappa U^{\Gamma'}_{ij}(J^{0ij}+\tfrac{1}{2}i\eta^{kli}J_{kl}{}^i)\}.$$

By substituting $\overset{\mathrm{L}}{\Psi}{}^i{}_j$ from (A2.6) and (A2.7) in place of $\Psi^i{}_j$ in this expression, we omit the contribution from free dynamical modes and the contributions to the constraint variables independent of the matter. Since $\psi^{\lambda\mu\rho\sigma}$ is to be the known Weyl tensor, this gives:

The second Mach condition (formal statement). A globally hyperbolic solution of the Einstein field equations (with zero cosmological constant) will satisfy the second Mach condition if

$$\lim_{\substack{S\to S_\infty\\ D^+(S_\infty)=M}}\Bigg\{\int_s(C^{\lambda\mu\rho\sigma}\nabla_0G^{\alpha'\beta'\gamma'\delta'}_{\lambda\mu\rho\sigma}-G^{\alpha'\beta'\gamma'\delta'}_{\lambda\mu\rho\sigma}\nabla_0C^{\lambda\mu\rho\sigma})\sqrt{h}\,dS^0-8$$
$$\times\mathscr{R}e\int_s\{\overset{\mathrm{L}}{\Psi}{}^{ij}(\partial_0U^{\Gamma'}_{ij}-2K^l{}_iU^{\Gamma'}_{lj})+iU^{\Gamma'}_{lj}\eta^{jkl}\mathscr{D}_k\overset{\mathrm{L}}{\Psi}{}^i{}_l$$
$$+\kappa U^{\Gamma'}_{ij}(J^{0ij}+\tfrac{1}{2}\eta^{klj}J_{kl}{}^i)\}\sqrt{h}\,dS^0\Bigg\} = 0$$

* It is not self-adjoint with respect to the hermitian conjugate, since $\tilde{R}_{ij}$ is not hermitian. A hermitian system can be obtained using a derivative Δ_k (instead of $\mathscr{D}_k$) defined by $\Delta_kV_i = V_{i|k}-\tfrac{1}{2}iK^l{}_{[j}\eta^k{}_{i]l}V_k$ and writing (A2.3) as $\Delta_k\Psi^k{}_i = \kappa J_i$. Note also that $h_{ik}\overset{\mathrm{T}}{\Psi}{}^k{}_j$ is not symmetric.

where the limit refers to a sequence of Cauchy surfaces S the limit of whose future Cauchy developments $D^+(S)$ is the whole space-time.

Further knowledge on the structure of singularities would be required for this limiting process to be made a well-defined operation in physically reasonable (singular) cosmological models, although for the known exact solutions there is a natural choice of a family of surfaces with respect to which the limit may be taken. The methods of investigation of the Machian character of particular models in Sections 6, 7 and 8 are essentially equivalent to a study of certain analyticity properties of the Green function $G^{\alpha'\beta'\gamma'\delta'}_{\lambda\mu\rho\sigma}$ for those cases.