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Modelling the *Pioneer* anomaly as modified inertia

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ABSTRACT

This paper proposes an explanation for the *Pioneer* anomaly: an unexplained Sunward acceleration of $8.74 \pm 1.33 \times 10^{-10}\,\mathrm{m\,s^{-2}}$ seen in the behaviour of the *Pioneer* probes. Two hypotheses are made. (1) Inertia is a reaction to Unruh radiation and (2) this reaction is weaker for low accelerations because some wavelengths in the Unruh spectrum do not fit within a limiting scale (twice the Hubble distance) and are disallowed: a process similar to the Casimir effect. When these ideas are used to model the *Pioneer* crafts' trajectories, there is a slight reduction in their inertial mass, causing an anomalous Sunward acceleration of $6.9 \pm 3.5 \times 10^{-10}\,\mathrm{m\,s^{-2}}$ which agrees within error bars with the observed *Pioneer* anomaly beyond 10 au from the Sun. This new scheme is appealingly simple and does not require adjustable parameters. However, it also predicts an anomaly within 10 au of the Sun, which has not been observed. Various observational tests for the idea are proposed.

Key words: gravitation – celestial mechanics – Solar system: general – cosmology: theory.

1 INTRODUCTION

Anderson et al. (1998) have detected a constant unexplained acceleration of both *Pioneer 10* and *Pioneer 11* of $8.74 \pm 1.33 \times 10^{-10}\,\mathrm{m\,s^{-2}}$ directed approximately towards the Sun. Since the behaviour of the *Pioneer* craft should be predictable because of their spin-stabilization (Anderson et al. 1998, 2002) but disagrees with our present understanding of motion, and since no convincing mundane physical explanation has so far been successful, the anomaly will be assumed here to be real.

Combining Newton's second law, and his law of gravity, the acceleration of a body of gravitational mass m_g due to a larger body of mass M at a distance r is

$$a = \frac{GMm_\mathrm{g}}{m_\mathrm{i}r^2}, \tag{1}$$

where m_i is the inertial mass and G is Newton's gravitational constant. Usually, we assume that $m_\mathrm{i} = m_\mathrm{g}$ (the equivalence principle). However, this formula shows that to account for the anomalous acceleration a of the *Pioneer* craft towards the Sun we can increase G, increase M, or increase $m_\mathrm{g}/m_\mathrm{i}$.

The *Pioneer* anomaly is similar to the galaxy rotation problem which also involves an unexplained acceleration towards a centre of mass. One solution to this problem was proposed by Milgrom (1983) and is called MOdified Newtonian Dynamics (MOND). This theory has proved successful in reproducing galaxy rotation curves and is usually (but not necessarily) based on the first approach mentioned above: G is increased for accelerations lower than $1.2 \times 10^{-10}\,\mathrm{m\,s^{-2}}$. This is also the approach of the relativistic extension of MOND by Bekenstein (2004) which is called TeVeS. As an alternative, G can be modified at long distances. This is the approach taken by the Scalar–Tensor–Vector Gravity (STVG) theory of Moffat and Brownstein (2006) which has been used to model the *Pioneer* anomaly, though they needed adjustable parameters to do this. The conformal gravity theory of Mannheim (1990) also modifies G so that it is repulsive at long distances.

An example of the second approach (increasing M) is the dark matter hypothesis of Zwicky (1933). Excess, invisible, matter is added to the galaxy to explain the implied extra centripetal acceleration. However, dark matter fits to galaxies have three free parameters, whereas MOND has only one: the mass-to-light ratio (M/L) (Sellwood 2004).

The third approach, reducing the inertial mass (m_i), was first suggested by Milgrom (1983) who realized that MOND could be explained as a modification of inertia instead of G. In later papers (Milgrom 1994, 1999), he suggested a possible physical cause for the inertial version of MOND which is discussed in Section 2.1 below. As he noted, there are some observations that imply that it is inertia that should be modified and not G or M. For example, the possible change in behaviour of the *Pioneer* craft upon moving from a bound to an unbound trajectory (to be confirmed, or not, soon, by the *Pioneer* team), and the planets, which are on bound orbits, do not seem to show the anomaly. Also, MOND behaviour in galaxies begins below a limiting acceleration and not beyond a limiting distance, as noted by Sanders & McGaugh (2002).

One possibility for a model of inertia is that of Haisch, Rueda & Puthoff (1994) who proposed that an accelerated object feels a magnetic Lorentz force through its interaction with a zero-point field (ZPF) similar to the Unruh field (Unruh 1976). This force is given by $F = -\Gamma\omega_\mathrm{c}^2\hbar a/2\,\pi c^2$, where Γ is the Abraham–Lorentz damping constant of the parton being oscillated, $\hbar$ is the reduced

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Planck constant, ω_c is the Compton scale of the parton below which the oscillations of the ZPF have no effect on it, c is the speed of light, and a is acceleration. Haisch et al. (1994) showed that this force behaves like inertia.

One objection to a modification of inertia is that it violates the equivalence principle, which has recently been tested to an accuracy of 10^{-13} kg by Baessler et al. (1999). However, this principle has not been tested at the low accelerations seen by the *Pioneer* craft or by stars at the edges of galaxies.

2 THE MODEL

2.1 Unruh radiation curtailed at the Hubble distance

After work by Hawking (1974), Unruh (1976) showed that a body with an acceleration a sees thermal radiation of temperature T where

$$T = \frac{\hbar a}{2\pi c k}, \tag{2}$$

where k is Boltzmann's constant. The dominant wavelength of this radiation (λ_m) is given by Wien's displacement law ($\lambda_m = W/T$), where W is Wien's constant. Replacing T using equation (2) and W with $\beta hc/k$, where $\beta = 0.2$ leaves

$$\lambda_m = \frac{4\pi^2 \beta c^2}{a}. \tag{3}$$

Milgrom (1994, 1999) realized that as the acceleration decreases the wavelength λ_m increases, and eventually becomes as large as the Hubble distance (c/H) where H is the Hubble constant. He speculated that at this point there would be a 'break in the response of the vacuum': the waves of Unruh radiation would be unobservable. He further speculated that this could have an effect on inertia, if inertia is linked to a form of Unruh radiation, as suggested by Haisch et al. (1994). He suggested this as a cause of MOND behaviour. Taking the limiting distance to be twice the Hubble distance (a Hubble diameter: $\Theta = 2c/H$), we can infer the acceleration at which this break would happen for Unruh radiation by rearranging equation (3) as

$$a = \frac{4\pi^2 \beta c^2}{\lambda_m}. \tag{4}$$

Substituting $\beta = 0.2, c = 3 \times 10^8\,\mathrm{m\,s^{-1}}$ and $\lambda_m = \Theta = 2c/H = 2.7 \times 10^{26}$ m (since $H = 2.3 \pm 0.9 \times 10^{-18}\,\mathrm{s^{-1}}$), the predicted critical acceleration is $a = 26 \times 10^{-10}\,\mathrm{m\,s^{-2}}$. Below this acceleration, inertia could be affected by Milgrom's break. This is larger than the acceleration constant of $a = 1.2 \times 10^{-10}\,\mathrm{m\,s^{-2}}$ required for MOND (Milgrom 1983) for fitting galaxy velocity curves. It is close to the *Pioneer* anomaly, but Milgrom's (abrupt) break cannot explain the *Pioneer* anomaly, since the *Pioneer* acceleration at 50 au from the Sun was still too large, about $10^{-5}\,\mathrm{m\,s^{-2}}$, and this acceleration implies Unruh wavelengths of only 0.03 per cent of the Hubble distance.

2.2 A Casimir-like effect at the Hubble scale

Here, Milgrom's long-wavelength cut-off idea is modified so that we assume that only wavelengths of the Unruh radiation that fit exactly into twice the Hubble distance ($\Theta = 2c/H$) are allowed: those harmonics with nodes at the boundaries. This is an idea similar to the Casimir effect in which the energy of the ZPF is reduced between conducting plates because only certain wavelengths can exist between them (Casimir 1948).

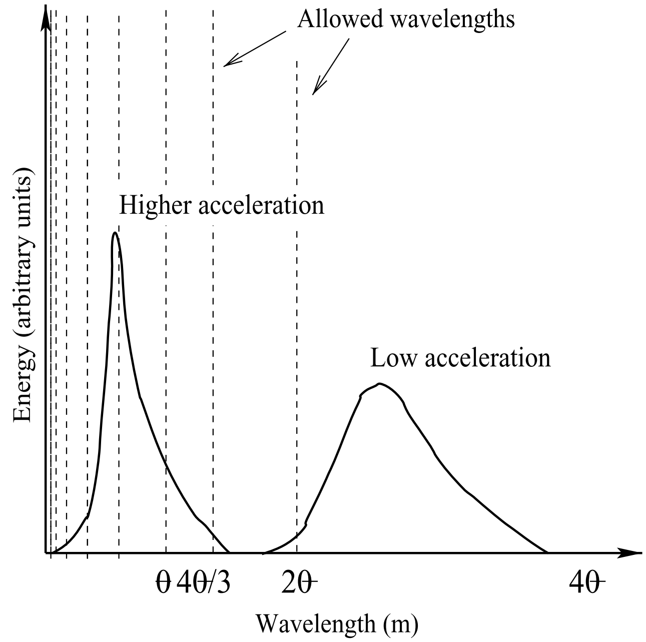

Figure 1. A schematic in which the vertical dashed lines show the wavelengths that fit within twice the Hubble distance and are allowed in this model. Unruh spectra for different accelerations are also shown. The one on the right-hand side represents a lower acceleration, and is more sparsely sampled by the allowed wavelengths.

Fig. 1 shows the energy of Unruh radiation as a function of the wavelength. The allowed wavelengths are shown by the dashed vertical lines. As for the Casimir effect, these wavelengths are given by

$$\lambda_n = \frac{2\Theta}{n}. \tag{5}$$

where n = 1, 2, 3…etc. For an object with high acceleration, the temperature of the Unruh radiation is high, the Unruh wavelengths seen are short and the Unruh energy spectrum looks like the curve on the left-hand side. In the schematic, this spectrum is sampled by five or six of the allowed wavelengths so that much of the energy in the Unruh spectrum remains. However, if the acceleration is reduced, then the object sees the spectrum on the right-hand side. In this case, only one of the wavelengths is allowed because the others do not fit within Θ and so the spectrum is more sparsely sampled, and the energy of the Unruh radiation is much lower than expected. In this new scheme, some spectral energy is lost at wavelengths shorter than Θ, and this allows the prediction of the *Pioneer* anomaly, which cannot be explained by the more abrupt *break* mentioned in Milgrom (1994, 1999) and discussed in Section 2.1.

If the Unruh energy spectrum is given by a function $f(\lambda)$, then the unmodified inertial mass (m_i) is assumed here to be proportional to the integral of this:

$$m_i \propto \int_0^\infty f(\lambda)\,\mathrm{d}\lambda. \tag{6}$$

To model the effect of the increasingly sparse sampling of the spectrum at long wavelengths, the weight of longer wavelengths in equation (6) is reduced by using a factor F to account for the reduction in sampling density when going from the continuous sampling of the spectrum to the discrete sampling. By direct calculation, it was found that the number of allowed wavelengths available to sample the Planck spectrum varied linearly as λ_m^{-1} over the range of wavelengths studied here, where λ_m is the peak wavelength of the spectrum (this was done by counting the number of allowed wavelengths where the spectral energy was more than 1 per cent of the peak energy). Therefore, we assume that $F = \frac{A}{\lambda_m} + B$, where A and B are constants. When $\lambda_m \to 0$, the normal continuous sampling should be recovered and $F = 1$. When $\lambda_m \to 4\Theta$, no energy is sampled so $F = 0$ (this is Milgrom's break, as discussed above). Using these conditions, A and B can be found and the factor can be shown to be $F = 1 - \lambda_m/4\,\Theta$. The model for the modified inertial mass (m_{I}) is therefore

$$m_{\mathrm{I}} \propto \int_0^{\infty} f(\lambda)\,\mathrm{d}\lambda \left(1 - \frac{\lambda_m}{4\Theta}\right). \tag{7}$$

From equations (6) and (7)

$$m_{\mathrm{I}} = m_{\mathrm{i}} \left(1 - \frac{\lambda_m}{4\Theta}\right). \tag{8}$$

Using equation (3) and assuming that the equivalence principle applies to the unmodified inertial mass, that is, $m_{\mathrm{i}} = m_{\mathrm{g}}$, the modified inertial mass m_{I} becomes

$$m_{\mathrm{I}} = m_{\mathrm{g}} \left(1 - \frac{\beta\pi^2 c^2}{a\Theta}\right). \tag{9}$$

Here, m_{I} behaves in a way similar to what would be expected for MOND (Milgrom 1983). For large accelerations, the second term in the brackets is negligible and the standard inertial mass is recovered. However, as the acceleration decreases, the second term becomes larger, and m_{I} falls farther below m_{g}. For accelerations much lower than that seen here, it is possible for the term in brackets to be negative, implying a negative inertial mass. However, in this model, such a low acceleration would never be attained, since a body with an inertial mass approaching zero would tend to accelerate again: there is a minimum acceleration. For an acceleration of 9.8 m s^{-2}, the inertial mass of a 1 kg object is predicted to be 7×10^{-11} kg lower. For the small accelerations seen by the *Pioneer* craft, which are far from a gravitational source, the inertial mass is predicted to decrease by 0.01 per cent. At some point, the acceleration, acting now on a lower inertial mass, increases again. Eventually a balance is achieved, as modelled below, around a particular acceleration. Assuming modified inertia, the equation of motion for the *Pioneer* craft is

$$F = m_{\mathrm{I}} a = \frac{G\,\mathrm{M}_{\odot} m_{\mathrm{g}}}{r^2}, \tag{10}$$

where $\mathrm{M}_{\odot}$ is the solar mass and r is the distance from the Sun. Substituting for m_{I} from equation (9), we can find the balance point mentioned above:

$$a = \frac{G\,\mathrm{M}_{\odot}}{r^2} + \frac{\beta\pi^2 c^2}{\Theta}. \tag{11}$$

Therefore, the acceleration is given by the usual Newtonian inverse square law, but with an additional constant term caused by the loss of inertia. This new term has a value of $6.9 \pm 3.5 \times 10^{-10}$ m s^{-2} which is about six times larger than the 1.2×10^{-10} m s^{-2} required for MOND. The 40 per cent (± 3.5) uncertainty arises because of uncertainties in the Hubble constant (see Section 2.1).

According to equation (11), all bodies, even if there is no source of gravity ($\mathrm{M}_{\odot} = 0$), would show a minimum acceleration, given by the second term on the right-hand side, which can be rearranged to give $\frac{1}{2}\beta\pi^2 cH \sim 0.99 \times cH$ which is close to the observed Hubble expansion rate (cH). Therefore,

$$a = \frac{G\,\mathrm{M}_{\odot}}{r^2} + 0.99 \times cH. \tag{12}$$

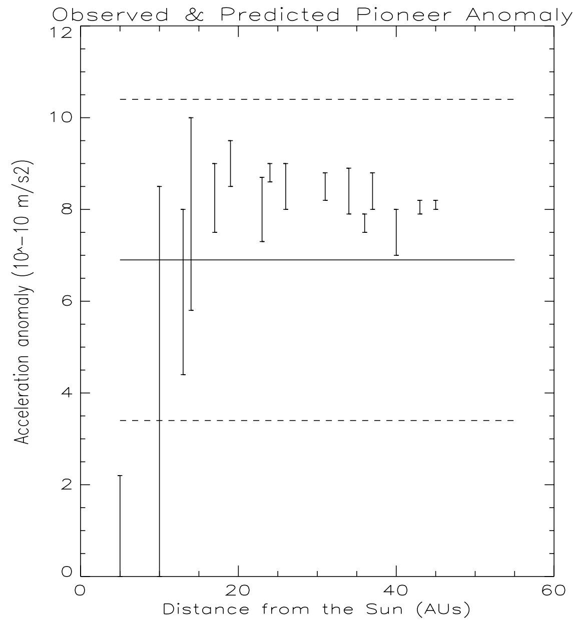

Figure 2. The bars show the observed *Pioneer 10* and *Pioneer 11* anomalies as a function of distance from the Sun (au) (taken from Anderson et al. 2002). The solid line shows the *Pioneer* anomaly predicted by equation (11) and the dashed lines represent the error bars for the model.

3 RESULTS

The vertical error bars in Fig. 2 show the observed *Pioneer* anomaly as a function of distance from the Sun out to 45 au taken from Anderson et al. (2002). Within about 10 au of the Sun, the anomaly was indistinguishable from zero. It increased after about 10 au to an approximately constant value of 8.74×10^{-10} m s^{-2}.

The solid line shows the acceleration anomaly predicted by the extra term in equation (11) and the horizontal dashed lines show the error bars for the prediction. The predicted anomaly was a constant $6.9 \pm 3.5 \times 10^{-10}$ m s^{-2}, which is in agreement with the observed anomaly from 10 to 45 au from the Sun.

The model predicts that the anomaly should also be found within 10 au of the Sun and this does not agree with the first data point at 6 au from Anderson et al.'s (2002) data (see the leftmost bar in Fig. 2) which shows no anomaly. Also, the planets do not show an anomaly. This difference may be due to the *Pioneer*'s unbound trajectory. As noted by Milgrom (2006), for theories of modified inertia the acceleration depends on the trajectory as well as the position. A further analysis of the *Pioneer* data is ongoing (Toth & Turyshev 2006) and should improve the data resolution at the crucial point where the *Pioneer*s' trajectories became unbound: between 5 and 10 au.

The fit of this model to the *Pioneer* data is less close than that obtained by Brownstein & Moffat (2006). However, they fitted their model to the *Pioneer* anomaly data using two adjustable parameters, whereas there are no adjustable parameters here.

4 DISCUSSION

One of the consequences of this idea, not considered by the parametrization of Section 2.2, is that at certain accelerations the Unruh spectral peak is directly sampled by the allowed wavelengths, and $m_{\mathrm{I}}/m_{\mathrm{g}}$ is then at a temporary peak. At other accelerations, the nearest sampling wavelength would be slightly off-peak and so

m_I/m_g would be lower. These ideas therefore predict that the *Pioneer* data (and also galaxy rotation curves) may show a radial variation in the ratio of m_I/m_g as the favoured accelerations are sampled one by one moving out from the Newtonian regime near the centre of the Solar system (or galaxy) to the lower accelerations farther out, with further consequences for dynamics. At 40 au from the Sun, the number of allowed wavelengths in the Unruh spectrum seen by *Pioneer*, counted as described in Section 2.2, is about 4000. Thus, the spectrum is still quite well sampled, and these variations may be too small to detect. However, near the edges of galaxies, accelerations are much lower, and the Unruh spectrum would be sampled by only a few wavelengths. Therefore, the differences in the ratio m_I/m_g between a case in which the discrete sampling hits the spectral peak and a case in which it misses it, would be more obvious, and the impact on stellar dynamics of the variations should be more easily detected.

As mentioned above, an analysis of newly recovered *Pioneer* data from the inner Solar system is currently in progress (Toth & Turyshev 2006) and would support a theory of modified inertia, though not necessarily this one, if it is confirmed that the anomaly began at the same time that the *Pioneer* probes moved from bound orbits to hyperbolic ones (Milgrom 1999). These new data may also resolve the direction of the anomalous force. An acceleration towards the Sun would imply modified G, one towards the Earth would imply a problem with time, and an acceleration along the *Pioneer* trajectory would imply some kind of modified inertia.

Zhao (2005) and Zhao & Tian (2006) have shown that if MOND is true instead of Newtonian theory, then Roche lobes should be more squashed and therefore it should be possible to test for MOND by investigating a local Roche lobe. This test could also differentiate between modified gravity and modified inertia versions of MOND, since for modified inertia the shape of the Roche lobe would depend on the approach trajectory of the probe, and for modified gravity it would not.

In this scheme, there is a minimum allowed acceleration which depends on a Hubble scale Θ, so, if Θ has increased in cosmic time, there should be a positive correlation between the anomalous centripetal acceleration seen in equivalent galaxies, and their distance from us, since the more distant ones are seen farther back in time when, if the universe has indeed been expanding, Θ was smaller. The M/L does seem to increase as we look farther away. The M/L of the Sun is 1 by definition, for nearby stars it is 2, for galaxies it is 50, for galaxy pairs it is 100 and for clusters it is 300. As an aside, equation (11) could be used to model inflation, since when Θ was small in the early universe the minimum acceleration is predicted to be larger.

Part of this scheme is the hypothesis that Unruh radiation of very low temperature is weaker than expected, because of a wavelength limit, so it is logical to extend this to the temperature of any object. If the limiting wavelength idea is correct, then the energy radiated by a very cold object should be less than that expected from the Stefan–Boltzmann law. The coldest temperature achieved so far is 100 pK at the Helsinki University of Technology (Knuuttila 2000). Using Wien's law, an object this cold would have a peak radiating wavelength of 3×10^7 m. By analogy to equation (8), the energy of the blackbody radiation spectrum (E) would be modified to E' as

$$E' = E\left(1 - \frac{\lambda_m}{4\Theta}\right) = E(1\text{–}2.7 \times 10^{-20})\,\mathrm{J}. \tag{13}$$

It is unknown to the author whether differences in radiating energy as small as this can be detected.

The Hawking (1974) temperature of a black hole is given by an expression very similar to that of Unruh but involving the mass of a black hole M:

$$T = \frac{\hbar c^3}{8\pi G M k}. \tag{14}$$

As in Section 2.1, we can use Wien's law ($T = W/\lambda_m = \beta hc/k\,\lambda_m$) again to substitute for T and impose a limit on the allowed wavelength:

$$\frac{16\pi^2 G M \beta}{c^2} \leqslant \frac{2c}{H}. \tag{15}$$

Therefore,

$$M \leqslant \frac{c^3}{8\pi^2 G \beta H}. \tag{16}$$

Substituting $c = 3 \times 10^8\,\mathrm{m\,s^{-1}}$, $G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$, $\beta = 0.2$ and $H = 2.3 \times 10^{-18}\,\mathrm{s^{-1}}$, we get $M \leqslant 1 \times 10^{52}$ kg. This is a predicted maximum mass of a black hole: about $10^{22}\,\mathrm{M_\odot}$.

The assumptions made in equations (6) and (7) have not individually been verified, but they do produce results similar to the *Pioneer* anomaly. A criticism of this scheme could be that the parametrization of the decrease in sampling density neglects subtle variations as the Unruh spectrum falls between allowed wavelengths, and these variations could be useful for testing the idea. The simple model developed here should ideally be replaced by a model that calculates m_I more directly, by sampling the Unruh spectrum discretely.

5 CONCLUSIONS

Two hypotheses were made. (1) Inertia is a reaction to Unruh radiation and (2) this reaction is weaker for low accelerations because some wavelengths in the Unruh spectrum do not fit within a limiting scale (twice the Hubble distance) and are disallowed: a process similar to the Casimir effect.

Using these ideas, the *Pioneer* acceleration anomaly was predicted to be $6.9 \pm 3.5 \times 10^{-10}\,\mathrm{m\,s^{-2}}$, which agrees within error bars, beyond 10 au from the Sun, with the observed value of $8.74 \pm 1.33 \times 10^{-10}\,\mathrm{m\,s^{-2}}$.

This scheme is appealingly simple, and does not require adjustable parameters. However, the model predicts an anomaly within 10 au of the Sun which is not observed. Various tests of this idea are also discussed, including the possibility that subtle variations in galaxy acceleration curves (if not the *Pioneer* data) might be detectable.

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REFERENCES

Anderson J. D., Laing P. A., Lau E. L., Liu A. S., Nieto M. M., Turyshev S. G., 1998, Phys. Rev. Lett., 81, 2858

Anderson J. D., Laing P. A., Lau E. L., Liu A. S., Nieto M. M., Turyshev S. G., 2002, Phys. Rev. D, 65, 082004

Baessler S., Heckel B. R., Adelberger E. G., Gundlach J. H., Schmidt U., Swanson H. E., 1999, Phys. Rev. Lett., 83, 3585

Bekenstein J. D., 2004, Phys. Rev. D, 70, 3509

Brownstein J. R., Moffat J. W., 2006, Class. Quantum Gravity, 23, 3427

Casimir H. B. G., 1948, Proc. Kon. Nederland. Akad. Wtensch., B51, 793
Eisberg R., Resnick R., 1985, Quantum Physics, 2nd edn. John Wiley and Sons Ltd, New York
Haisch B., Rueda A., Puthoff H. E., 1994, Phys. Rev. A, 49, 678
Hawking S., 1974, Nat, 248, 30
Knuuttila T. A., 2000, DSc thesis, Helsinki University of Technology, http://lib.tkk.fi/Diss/2000/isbn9512252147/
Mannheim P. D., 1990, Gen. Relativ. Gravit., 22, 289
Milgrom M., 1983, ApJ, 270, 365
Milgrom M., 1994, Ann. Phys., 229, 384
Milgrom M., 1999, Phys. Lett. A, 253, 273
Milgrom M., 2005, in Mamon G., Combes F., Deffayet C., Fort B., eds, EAS Publications Series Vol. 20, Mass Profiles and Shapes of Cosmological Structures. EdP Sciences, Les Ulix Cedex A, p. 217
Sanders R. H., McGaugh S. S., 2002, ARA&A, 40, 263
Sellwood J. A., 2004, in Ryder S., Pisano D. J., Walker M., Freeman K. C., eds, IAU Symp. 220, Dark Matter in Galaxies. Astron. Soc. Pac., San Francisco, p. 27
Toth V. T., Turyshev S. G., 2006, preprint (gr-qc/0603016v1)
Unruh W. G., 1976, Phys. Rev. D, 14, 870
Van Allen J. A., 2003, Am. J. Phys., 71, 448
Zhao H., 2005, A&A, in press (astro-ph/0511713)
Zhao H., Famaey B., 2006, ApJ, in press (astro-ph/0512425)
Zhao H., Tian L., 2006, A&A, 450, 1005
Zwicky F., 1933. Helv. Phys. Acta, 6, 110

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