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Arnold Shapiro's Eversion of the Sphere

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Dedication. We dedicate this article to the memory of Arnold Shapiro, who gave the first example of how to turn the sphere inside out, but never published it. His is not the simplest, nor the most interesting of the many explicit eversions that have been devised since. It is, however, the only one that uses only standard topological constructions. Thus it is of value to the history and philosophy of mathematics, for, had it been better explained in its day, the subject would hardly have occupied so many people in the intervening two decades. We have written it up from memory and in a conventional expository style, but hope that the illustrations will aid the inner eye to see Shapiro's ingenious ideas.

An *immersion* $f: M^2 \to \mathbb{R}^3$ of a closed surface in space is a smooth map whose differential is everywhere of maximal rank. A one-parameter deformation, $f_t : M^2 \to \mathbb{R}^3$, $t \in [0, 8]$, is a *regular* homotopy if $t \to f_t$ is continuous in the C^1-topology on the space of immersions. If f is one-to-one, it is called an *embedding* and a regular homotopy of embeddings is called an *isotopy*. An *eversion* of the sphere is a regular homotopy from the standard embedding $f_0 : S^2 \to \mathbb{R}^3$ to $f_8(x, y, z) = f_0(-x, -y, -z)$. Hilbert's student, Werner Boy, recognized the importance of regular homotopies in his dissertation of 1901. One dimension lower, he discovered that the tangent winding number classifies immersions of the circle in the plane up to regular homotopy [1]. In 1937, Whitney and Graustein [13] found an elegant proof of this theorem. The two dimensional problem was not resolved until the late fifties, when S. Smale proved the remarkable fact that all immersions of the sphere in space are regularly homotopic [12]. In particular, there must exist an eversion of the sphere. Bott asked to be shown an explicit geometrical construction of an eversion. One way, suggested by H. Hopf and sketched by N. Kuiper [5], would be to design a regular homotopy taking the standard sphere to a double cover of Boy's surface [2]. This surface, an immersed real projective plane, has one curve of self-intersection with one triple point. In December of 1960, the late Arnold Shapiro explained such a motion to B. Morin. In this note we would like to share this ingenious eversion with the reader.

In the intervening years many interesting eversions have been constructed using a variety of graphical methods. A. Phillips [11] designed a series of drawings based on the level curves of Boy's surface and the deformation of its double cover to an embedded sphere. A new idea, suggested by M. Froissart, was adopted by Morin in 1967 to create an eversion that does not use Boy's surface. The principal stages of an eversion through Morin's surface were fashioned out of wire mesh by C. Pugh. A masterpiece of computer graphics, based on these models, was programmed by Nelson Max [6]. Humbler graphical methods were used by J. Petit [8, 9] to display Morin's generic eversion, emphasizing the evolution of the double locus. A countable sequence of symmetric eversions through generalized Boy and Morin surfaces were presented topologically by G. Francis [3, 4]. Analytic parametrizations of such eversions were given by Morin [7]. A more complete survey of the subject appears in [10]. But Shapiro's original conception differs in one important aspect from all of these. The subsequent examples were meant to evoke complete mental pictures of every part of the motion, and for this, novel expository techniques were required. Shapiro assembled standard pieces of differential topology as they were current in the fifties. Perhaps only a deficiency in illustration prevented Shapiro's vision from gaining the recognition it certainly deserved.

His strategy is to modify a torus to an immersed sphere in three different ways, so that successive modifications span (in a sense to be made precise) the form of a canonical deformation, Shapiro's *baseball move,* which connects them by a regular homotopy. The first of these is easily seen to be homotopic to the sphere and the third to the double cover of Boy's surface. Let $\delta : D^2 \to \mathbb{R}^3$ denote an immersion of the closed disc in space so that $\gamma = \delta \,|\, \partial D^2 : S^1 \to T^2$ is an embedding of the circle in the torus, and so that the disc is normal to the torus along γ. (We shall call the image of a map by the same name as the map if no ambiguity arises.) We may thicken δ to an immersion $\bar{\delta} : D^2 \times I \to \mathbb{R}^3$, $I = [-1, +1]$, so that $\bar{\gamma} = \bar{\delta} \,|\, \partial D \times T^2$ is an embedded ribbon on the torus. If γ is nontrivial, in the sense that the complement of the interior of $\bar{\gamma}$ in T^2 is an embedded annulus A, then its union with the two discs $\bar{\delta} \,|\, D^2 \times \{\pm 1\}$ is a piecewise immersed sphere with corners along two curves parallel to γ. We smoothe these corners according to the following model of a cross section of the corner. Consider the plane curve formed by two rays separated by an angle λ, $0 < \lambda \leqslant \pi$, parametrized in polar coordinates as $r = s^2$, $\theta = \lambda(1 + \mathrm{sgn}(s))/2$. The modification $r' = s^2 + \epsilon$, $\theta' = \lambda(1 + \sigma(s))/2$, where σ is a smooth sigmoid approximation to the signum which rises from -1 for $s \leqslant -\epsilon$ to $+1$

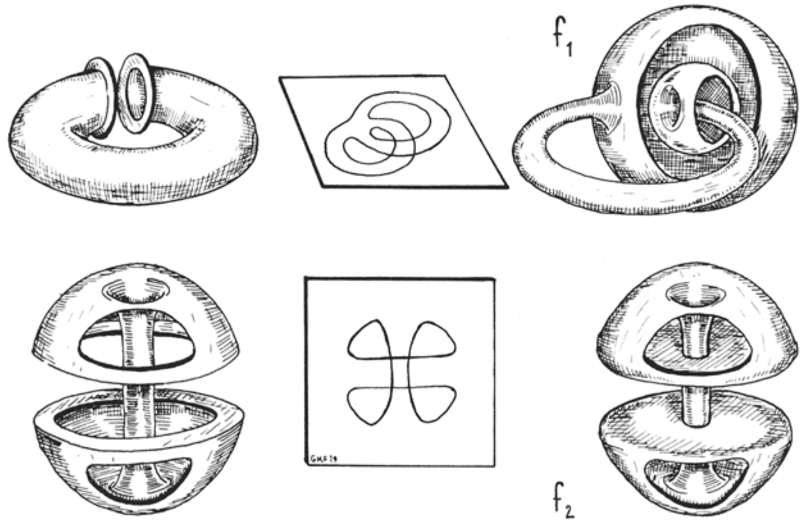

Figure 1. Modifying the torus to two regularly homotopic immersed spheres by attaching discs to meridian & equatorial bands on the torus

for $\epsilon \leqslant s$ and $\sigma'(s) > 0$ for $-1 < s < +1$, smoothes the corner in a small neighborhood whose size is controlled by $\epsilon > 0$. Note that if λ varies independently with time, this construction also defines a regular homotopy between the smoothed curves. This feature will be important later.

Modification by a meridian disc on the inside of the torus, or by a disc attached to a comeridian (latitude) on the outside, e.g. the equatorial disc spanning the hole of the donut, produces an embedded sphere. Shapiro did just the opposite. Attach a disc to a meridian on the outside of the torus. The resulting immersion $f_1 : S^2 \to \mathbb{R}^3$ consists in two disjoint spherical shells connected by a toroidal tunnel from the inside of the inner shell to the outside of the outer shell, see Figure 1. Attach a horizontal disc to the outside equator of the torus and the resulting immersion $f_2 : S^2 \to \mathbb{R}^3$ consists in two disjoint spherical shells connected by a vertical tunnel from the north pole of the upper shell to the south pole of the lower shell. To minimize the graphical ambiguities caused by drawing double curves, we use the device of *windows* in the figures. These are transparent discs embedded in the surface as well as in the viewing plane under projection. They are meant to reveal detail that would otherwise be invisible from the outside. The equatorial section of f_1 and the polar section of f_2 are given for reference. Figure 2 suggests a regular homotopy from the standard sphere f_0 to f_1. But for a homotopy from f_1 to f_2 we shall need Shapiro's *baseball move.*

Suppose that two modifications by $(\delta_i, \gamma_i), i = 1, 2$, are situated so that γ_1 and γ_2 cross transversally at one point on T^2. Suppose further that at the patch common to both ribbons $\overline{\gamma}_1, \overline{\gamma}_2$, the thick discs $\overline{\delta}_1, \overline{\delta}_2$, extend locally on opposite sides of the torus. We can make a model for the domains of $\overline{\delta}_1$ and $\overline{\delta}_2$ by imagining two coins balanced at right angles, one on top of the other, with their edges fused along a small square, Figure 3. The corners on the fused coins K form one continuous simple closed curve κ that is reminiscent of the seam of a baseball, pinched together near the square. This curve κ supports two obvious embedded discs, which are isotopic to each other by a deformation through the interior of K. The spanning disc turns through an angle of 90° along κ during the isotopy. The image of this motion under the immersions $\overline{\delta}_1$ and $\overline{\delta}_2$, smoothed at the corners as prescribed earlier, gives the regular homotopy from f_1 to f_2.

Now consider a third simple closed curve γ_3 on T^2 which crosses the outside equator γ_2 once and which spans a Möbius band μ on the inside of the torus. A so-called (2, 1)-curve is an example. Since γ_3 also spans an, albeit

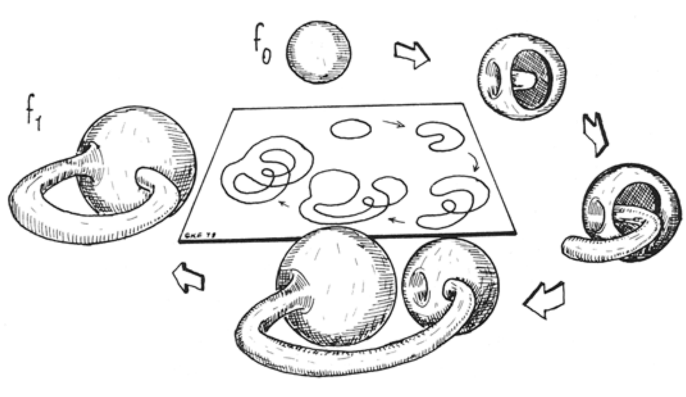

Figure 2. Regular homotopy from a sphere to an immersed sphere; the cross section is a regular homotopy in the plane

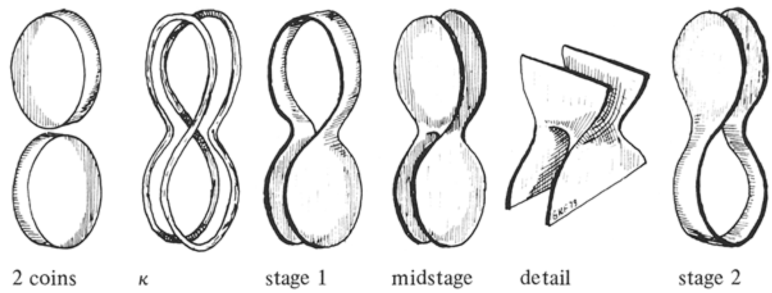

Figure 3. Shapiro's "Baseball move"

Figure 4. The (2, 1) torus curve spans a Möbius band whose complement retracts to a double cover to the Möbius band

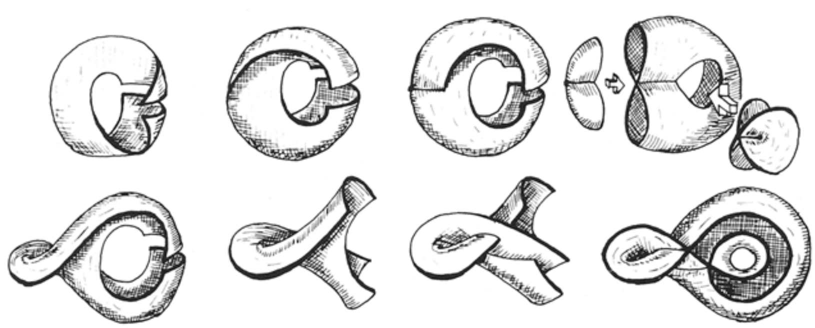

Figure 5. Two ways of closing a Möbius band by attaching a singular disc

immersed, disc δ_3 on the outside of the torus, it separates an immersed projective plane, $\beta : P^2 \to B^2 \subset \mathbb{R}^3$, into two pieces. Although we shall reconstruct the disc δ_3 de novo, its existence may be inferred as follows. Boy's surface B^2 contains an embedded, unknotted Möbius band with only 1/2 twist in it. Thus it can be encased in an embedded, unknotted torus T^2 which cuts B^2 along a (2, 1)-curve on T^2. The complement of the Möbius band is the disc we seek. Now modify T^2 by means of δ_3 to obtain the immersion $f_3 : S^2 \to \mathbb{R}^3$ which is regularly homotopic to f_2 by a second baseball move.

We next define a regular homotopy from f_3 to a double covering of B^2. The (2, 1)-band $\overline{\gamma}_3$ spans a thickened Möbius band $\overline{\mu}$. The (closure of the) complement of $\overline{\mu}$ in the solid torus is a twisted embedded solid semi-torus (semi-disc x S^1). The retraction of a semicircle to its diameter models the regular homotopy of the toral part of f_3 to the annulus parallel to μ on the boundary of $\overline{\mu}$. Smoothing the 90° corners produces an immersion parallel to Boy's surface. Now slide this surface along normals to B^2 to obtain the immersion $f_4 : S^2 \to B^2$. An easily overlooked problem remains to be resolved. There is no reason to expect f_4 to map antipodes of S^2 into the same point on B^2 as the standard parametrization does. Since $\beta : P^2 \to B^2$ is an immersion in general position, it is not hard to imagine f_4 lifting to a double cover $\pi : S^2 \to P^2$. Because S^2 is the universal covering of the projective plane, there is a (sense preserving) diffeomorphism $h : S^2 \to P^2$ so that $\pi \circ h : S^2 \to P^2$ is the canonical double cover π_0 obtained by identifying antipodes. But h is isotopic to the identity. Hence we change the homotopy from f_3 to f_4 so that the latter is the canonical double covering $\beta \circ \pi_0 : S^2 \to P^2 \to B^2 \subset \mathbb{R}^3$. Reflecting the homotopy f_t from f_0 to f_4 by setting $f_{8-t} = f_t \circ \alpha$, where $\alpha : S^2 \to S^2$ is the antipodal map, completes Shapiro's eversion of the sphere.

While the first baseball move, from f_1 to f_2, may be imagined, if not exactly visualized since the immersed coins cross each other, the second move, f_2 to f_3, defies the imagination without a good view of Boy's surface. The mental difficulty of continuing the Möbius band spanning the (2, 1)-curve along a disc may be appreciated from the following example. A naive attempt of closing up the border, Figure 5, produces Steiner's crosscap. Although the two pinchpoints (Whitney umbrellas) can be cancelled, the double curve must first be made to cross itself at a triple point. In our second attempt, Figure 5, we avoid the

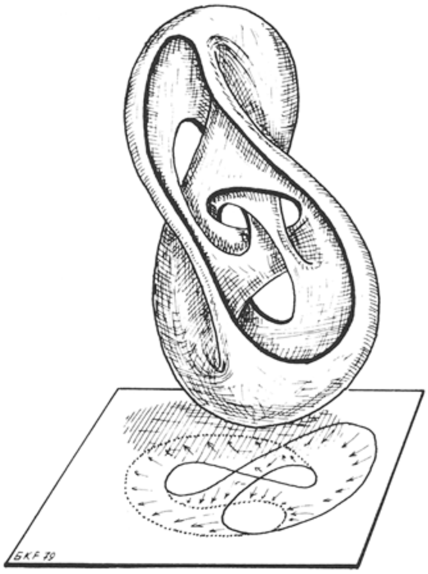

Figure 6. Boy's surface with four windows, generated from an immersed Möbius band and an embedded disc by a regular homotopy of cross sections from just above the saddle to just below the cusp

outside pinchpoint by introducing an apparent cusp. We next bring the entire border to the level of a plane above the, now immersed, Möbius band. Parametrizing this curve as a plane immersion of the circle, $\varphi : S^1 \to \mathbb{R}^2$, we note that it has the same tangent winding number as an embedded circle. By the Boy-Whitney-Graustein theorem, there is a regular homotopy $\varphi_t : S^1 \times I \to \mathbb{R}^2$ between them. One such homotopy is suggested by the arrows in the plane at the base of Figure 6. If we elevate this homotopy $S^1 \times I \to \mathbb{R}^2 \times \mathbb{R}^1 : (s, t) \to (\varphi_t(s), t)$, we obtain an immersed cylinder. Its double locus, with its triple point, may be imagined by closing the four windows. Now cap the cylinder with a disc to complete one view of Boy's surface.

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