c_p = specific heat at constant pressure of gas
k = coefficient of thermal conductivity
k_1 = constant of proportionality for Stokes flow
K = constant of order unity in defining Nu
l = characteristic dimension of dust particle
m = mass flow of gas
n = mass flow of dust
Nu = nusselt number
p = gas pressure
R = gas constant
T = gas temperature
u = gas velocity
v = dust velocity
α = ratio of specific heats of gas
λ = mass of individual dust particle
μ = viscosity of gas
ρ = gas density
τ = dust temperature

Subscripts

0 = free-stream conditions
1 = point immediately behind shock
2 = point downstream of shock where gas and dust are at identical velocities and temperatures
n = normal component of velocity
t = tangential component of velocity

The flow of a dusty gas through a shock wave has been analyzed by G. F. Carrier,[1] who considered a homogeneous mixture of a perfect gas and small particles.

Carrier's analysis considered the conservation equations, equation of state for the gas, and the energy-transfer relations as:

$$mu + nv + p = \text{constant} \tag{1}$$

$$(mu^2/2) + (nv^2/2) + mc_pT + nc\tau = \text{constant} \tag{2}$$

and when $m = \rho u = \text{constant}$, $pu = mRT$ (3)

$$\lambda v(dv/dx) = 1/2C_Dl^2\rho(u - v)^2 \tag{4}$$

$$\lambda cv(d\tau/dx) = Nu\ k(T - \tau)l \tag{5}$$

The ratio of Eqs. (4) and (5) gave

$$(d\tau/dx)/(dv/dx) = \beta[(T - \tau)/(u - v)] \tag{6}$$

where $\beta = 2Kk/c\mu$ when $Nu \simeq KC_DRe$. Since Eqs. (1), (2), and (3) define T and τ as functions of u and v, Eq. (6) can be written

$$d\tau(u, v)/dv = \beta\{[T(u, v) - \tau(u, v)]/(u - v)\} \tag{7}$$

Eq. (7) can then be rearranged into the form

$$du/dv = [\beta f_1(u, v) - f_2(u, v)]/[f_3(u, v)] \tag{8}$$

When the conditions just behind the shock wave are taken as the initial values $u = u_1$, and $v = u_0$, Eq. (8) may be numerically integrated to determine $u(v)$; similarly, $x(v)$ may be determined by integration of Eq. (4).

Carrier's method may be applied to the two-dimensional oblique shock-wave flow if it is assumed that the shock is unaffected by the dust and if the flow downstream is taken as $u^2 = u_n^2 + u_t^2$ and $v^2 = v_n^2 + v_t^2$. The dust-particle drag equation may then be written for the normal and tangential components of flow as:

$$\lambda v_t(dv_t/dy) = 1/2[C_D]_t\rho l^2(u_t - v_t)^2 \tag{4a}$$

$$\lambda v_n(dv_n/dx) = 1/2[C_D]_n\rho l^2(u_n - v_n)^2 \tag{4b}$$

Assuming that the flow is in the Stokes region and $C_D = 24/Re$, the ratio of Eqs. (4a) and (4b), letting $v_t = dy/dt$ and $v_n = dx/dt$, gives

$$\frac{v_n(dv_n/dx)}{v_t(dv_t/dy)} = \frac{dv_n}{dv_t} = \frac{u_n - v_n}{u_t - v_t} \tag{9}$$

The heat-transfer equation may be written

$$\lambda c(d\tau/dt) = Nu\ kl(T - \tau) \tag{10}$$

Taking ratios with Eqs. (4a) and (4b)

$$\frac{1}{v_t}\frac{(d\tau/dt)}{(dv_t/dy)} = \frac{2Kk}{c\mu}\frac{(T - \tau)}{(u_t - v_t)} \tag{11a}$$

$$\frac{1}{v_n}\frac{(d\tau/dt)}{(dv_n/dx)} = \frac{2Kk}{c\mu}\frac{(T - \tau)}{(u_n - v_n)} \tag{11b}$$

These Eqs. then reduce to

$$d\tau/dv_t = \beta[(T - \tau)/(u_t - v_t)] \tag{12a}$$

$$d\tau/dv_n = \beta[(T - \tau)/(u_n - v_n)] \tag{12b}$$

By solving Eqs. (1), (2), and (3) for τ and differentiating, we may obtain

$$d\tau/dv_n = \xi[d(u^2)/dv_n] + \phi \tag{13a}$$

$$d\tau/dv_t = \xi[d(u^2)/dv_t] + \psi \tag{13b}$$

when ξ, ϕ, and ψ are functions of u and v.

Eqs. (12a) and (12b) may then be written

$$\frac{d(u^2)}{dv_n} = \frac{1}{\xi}\left[\beta\frac{(T - \tau)}{(u_n - v_n)} - \phi\right] \tag{14a}$$

$$\frac{d(u^2)}{dv_t} = \frac{1}{\xi}\left[\beta\frac{(T - \tau)}{(u_t - v_t)} - \psi\right] \tag{14b}$$

When the turning angle of the flow through the shock and the free-stream conditions are known, then a set of initial conditions for the flow immediately behind the shock is determined since $[u_n]_0 = [v_n]_1$ and $[u_t]_0 = [v_t]_0 = [u_t]_1 = [v_t]_1$ and from Prandtl's equation

$$[u_n]_1[u_n]_0 = \sqrt{\frac{2\alpha}{\alpha + 1}RT_0 - \frac{\alpha - 1}{\alpha + 1}[u_t]_0^2} \tag{15}$$

If a relation between u_n and u_t is known or assumed for the flow behind the shock, then Eqs. (14a) and (14b) may be solved by numerical integration to obtain $u(v_n)$ or $u(v_t)$. Subsequently, $v_n(x)$ and $v_t(x)$ may also be obtained by a similar process using equations (4a) and (4b). The step-by-step numerical values obtained in the integration process then constitute an approximate description of the dusty-gas flow behind an oblique shock wave.

Reference

[1] Carrier, G. F., *Shock Waves in a Dusty Gas*, J. Fluid Mech., Vol. 4, Pt. 4, pp. 376–382, Aug. 1958.

Constant-Electric-Field and Constant-Magnetic-Field Magnetogasdynamic Channel Flow

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Introduction

Recently considerable interest has been shown in the investigation of electrode boundary-layer behavior in crossed-field accelerators, and simple closed-form solutions for the free-stream flow have been desired. Kerrebrock and Marble[1] obtained a simple solution for the case of constant-temperature constant-electric-field flow, which led to nearly-similar boundary-layer equations.[2]

Resler and Sears[3] and Wood and Carter[4] obtained other solutions to the free-stream cases, but these were difficult to adapt for boundary-layer work because the velocity variation could not be obtained explicitly as a function of the axial coordinate. A further difficulty with these solutions is that they require axially varying electric or magnetic fields.

This note develops a simple closed-form solution for a constant-electric- and constant-magnetic-field configuration. The solution allows simple interpretation of the free-stream behavior.

Analysis

The equations for the axial variation of velocity and Mach number of Ref. 3 may be written, for the case of constant area, in the form

$$\frac{dV}{dx} = \frac{1}{M^2 - 1}\frac{\sigma B^2}{(\dot{m}/\gamma)(V/M^2)}(V_3 - V)(V - V_1) \qquad (1)$$

$$\frac{dM}{dx} = \frac{1 + [(\gamma - 1)/2]M^2}{M^2 - 1}\frac{\sigma B^2}{(\dot{m}/\gamma)(V^2/M^3)}(V_3 - V)(V - V_2) \qquad (2)$$

whence

$$\frac{dV}{dM} = \frac{1}{1 + [(\gamma - 1)/2]M^2}\frac{V}{M}\frac{V - V_1}{V - V_2} \qquad (3)$$

Here, as in Ref. 3, V is the velocity, M the Mach number, $\dot{m}$ the mass flow per unit area, γ the ratio of specific heats, σ the electrical conductivity, B the applied magnetic field, and V_1, V_2, and V_3 are characteristic velocities described in terms of the electric field, magnetic field, and Mach number by

$$V_1 = \frac{\gamma - 1}{\gamma}\frac{E}{B}, \quad V_2 = \frac{1 + \gamma M^2}{1 + [(\gamma - 1)/2]M^2}\frac{V_1}{2}, \quad V_3 = \frac{E}{B}$$

The assumption of constant electric and magnetic fields gives V_1 and V_3 constant, so that Eq. (3) may be written

$$2\left(\frac{V}{V_1} - \frac{V_2}{V_1}\right) d\frac{V}{V_1} = \frac{V}{V_1}\left(\frac{V}{V_1} - 1\right)\frac{dM^2}{M^2}\frac{1}{1 + [(\gamma - 1)/2]M^2}$$

Noting that $2V_2/V_1$ may be written $\dfrac{[(\gamma + 1)/2]M^2}{1 + [(\gamma - 1)/2]M_2} + 1$,

this becomes

$$\left[2\frac{V}{V_1} - 1\right]\left[\frac{1}{M^2} + \frac{\gamma - 1}{2}\right] d\frac{V}{V_1} + \frac{V}{V_1}\left(\frac{V}{V_1} - 1\right)\left[\frac{-1}{M^4}\right]dM^2 = \frac{\gamma + 1}{2} d\frac{V}{V_1}$$

from which

$$d\left[\frac{V}{V_1}\left(\frac{V}{V_1} - 1\right)\left(\frac{1}{M^2} + \frac{\gamma - 1}{2}\right)\right] = \frac{\gamma + 1}{2} d\frac{V}{V_1}$$

This may be integrated directly to give

$$\frac{1}{M^2} + \frac{\gamma - 1}{2} = \frac{\gamma + 1}{2}\frac{V/V_1 - C}{V/V_1(V/V_1 - 1)} \qquad (4)$$

The constant of integration C determines the region of flow to which the solution applies, and a particularly simple solution is obtained by taking the line which passes through the singular point $V = V_1$, $M = 1$. In this case $C = 1$, so that we have

$$\frac{V}{V_1} = \frac{\gamma + 1}{2}\frac{M^2}{1 + [(\gamma - 1)/2]M^2} \qquad (5)$$

Substitution of this result into Eq. (1) gives

$$\frac{d(V/V_3)}{dx} = \frac{2\gamma}{\gamma + 1}\frac{\sigma B^2}{\dot{m}}\left(1 - \frac{V}{V_3}\right)$$

So that upon integration we find

$$V/V_3 = 1 - (1/\gamma)e^{-(\sigma B^2/\dot{m})[2\gamma/(\gamma + 1)]x} \qquad (6)$$

The other flow properties now follow easily, to give:

$$P/P_0 = \frac{1}{2}\left[1 + e^{-(\sigma B^2/\dot{m})[2\gamma/(\gamma + 1)]x}\right]$$

$$\rho/\rho_0 = [(\gamma - 1)/\gamma][1 - (1/\gamma)e^{-(\sigma B^2/\dot{m})[2\gamma/(\gamma+1)]x}]$$

$$\frac{T}{T_0} = 1 + \left[\frac{2 - \gamma}{2(\gamma - 1)} + \frac{1}{2}e^{-(\sigma B^2/\dot{m})[2\gamma/(\gamma+1)]x} - \frac{1}{2(\gamma - 1)}e^{-(\sigma B^2/\dot{m})[4\gamma/(\gamma+1)]x}\right]$$

$$j = (\sigma E/\gamma)e^{-(\sigma B^2/\dot{m})[2\gamma/(\gamma+1)]x}$$

These simple relations allow easy interpretation of the property variations in the accelerator. An interesting result is given by the ratio of Joule heating to electric energy input in the free stream, which is

$$\frac{\int_0^\infty j^2/\sigma dx}{\int_0^\infty jE dx} = \frac{1}{2\gamma}$$

This compares with the value $1/(2\gamma - 1)$ found for the "maximum acceleration" case of Ref. 3.

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Effect of Porosity on the Two-Dimensional Supersonic Sail

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The analysis by Daskin and Feldman[1] of the two-dimensional hypersonic sail was extended by the present author[2] to include the effect of the Busemann centrifugal correction to the Newtonian pressure law. The author followed this by a study of a two-dimensional supersonic sail.[3] In each case the sail was considered impermeable. The question naturally arises as to the effect of porosity on the performance of these sails. For the supersonic sail at least the correction is easily calculated as follows.

The pressure difference across the sail (see Fig. 1 for notation) is given by

$$C_{p_L} - C_{p_U} = k\theta_f \qquad (1)$$

where θ_f is the local flow deflection, and

$$k = 4/\sqrt{M_\infty^2 - 1} \qquad (2)$$

With an impermeable sail, $\theta_f = \theta$.

The porosity allows a flow through the sail from the under to the upper surface, which reduces the effective incidence of each element of the sail so that the overall lift is reduced. For the upward velocity v normal to the surface of the porous sail we may write

$$v/U = \sigma[(p_L - p_U)/q_\infty] \qquad (3)$$

where σ is a parameter describing the porosity. For the impermeable sail, $\sigma = 0$.

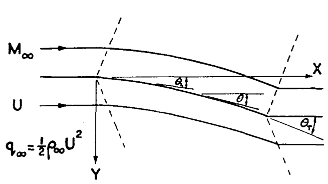

Fig. 1.