where

$$w = \text{resonance width } (D/3k_0)^{1/3}. \qquad (8.7)$$

It is a simple matter to derive the three-dimensional version of these equations. We have purposely kept the analysis simple and somewhat approximate so as not to obscure the underlying physics.

To be sure, the considerations of Secs. VI–VIII are very approximate. However, the purpose is to show in a simple way the origin and the nature of trapping phenomena within the context of Eqs. (5.2) and (6.5). Obviously, it would be desirable to have a more accurate treatment of these two equations, as well as mode coupling effects contained in the nonlinear terms of (5.1).

ACKNOWLEDGMENT

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Magnetohydrodynamic Channel Flows with Nonequilibrium Ionization

ARTHUR SHERMAN
General Electric Space Sciences Laboratory, Valley Forge, Pennsylvania
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The behavior of a magnetohydrodynamic channel, with finite segmented electrodes, when a plasma in a nonequilibrium state and with tensor conductivity flows through it with constant (uniform) velocity is presented. The nonequilibrium effect is taken into account by assuming the plasma electrical conductivity to be a linear function of the current density. The problem, which is highly nonlinear, is solved by expanding in powers of the proportionality constant to obtain a recursive set of linear equations. In this paper only the zeroth- and first-order solutions are presented. They are obtained numerically. For the cases studied, no shorting due to leakage between adjacent electrodes exists. Despite this inclusion of the nonequilibrium effect causes a reduction of the Hall (axial) electric field.

I. INTRODUCTION

MANY practical situations exist in which it would be desirable to have a nonequilibrium plasma condition in a magnetohydrodynamic channel flow.[1] In general, however, such a nonequilibrium condition is most easily created in a low-pressure plasma (less than one atmosphere) with a strong magnetic field (high induced electric field $\sim uB$), so that the Hall effect must be allowed for. As a result, a Hall current will flow axially unless the electrodes are segmented and an axial Hall voltage established. When the segments are finite there is an interaction between the extent of segmentation and the nonequilibrium effect due to the current crowding into one corner of the electrode. It will be the objective of the present work to analyze this current distribution when the electrodes are finite and nonequilibrium ionization is a factor.

A number of authors have analyzed the current distribution in a magnetohydrodynamic channel with constant σ and nonuniform geometry. The end currents occurring at the channel entrance or exit were among the first to be considered; the scalar conductivity case being solved first,[2] and the tensor conductivity case being treated later.[3,4] Then, as it was recognized that the Hall effect would require electrode segmentation, this problem was also treated, again for the tensor conductivity case.[5] In all cases, the flow velocity was assumed to be constant as was the scalar electrical conductivity and the Hall parameter $\omega\tau$ (where ω is the electron cyclotron frequency and τ is the mean time between electron–heavy-particle collisions). As a result, the equation to be solved in each case was Laplace's equation and the problem was a linear one, although in some cases the boundary condition was an unusual one.

In order to include nonequilibrium ionization in

[1] G. W. Sutton and A. Sherman, *Engineering Magnetohydrodynamics* (McGraw–Hill Book Company, Inc., New York, 1965).

[2] G. W. Sutton, H. Hurwitz, Jr., and H. Poritsky, Trans. AIEE Commun. Electron. **80**, 687 (1962).

[3] B. Podolsky and A. Sherman, J. Appl. Phys. **33**, 1414 (1962).

[4] G. W. Sutton, J. Appl. Phys. **34**, 396 (1963).

[5] H. Hurwitz, Jr., R. W. Kilb, and G. W. Sutton, J. Appl. Phys. **32**, 205 (1961).

any of the above problems, we have proposed to still use the single-fluid theory but now allow the scalar electrical conductivity to be a prescribed function of the absolute magnitude of the current density.[6] The Hall parameter is still assumed constant. Following this formulation, we found the formerly linear problem to be highly nonlinear but were able to solve for the Hartmann-type flow with Hall effect and nonequilibrium ionization.[6] The nonlinearity introduced becomes more severe in the case presently being considered, however, since partial differential equations are involved when channel nonuniformities are included.

Basically, the problem is treated by a perturbation analysis where the plasma conductivity σ is assumed to vary linearly with $|j|$. The proportionality constant is then assumed small and the solution expanded in powers of it. This results in a series of linear partial differential equations (Laplace's, Poisson's) which are solved in turn. The solutions are developed numerically due to the unusual boundary conditions required by the Hall effect. Only the zeroth- and first-order cases are carried out in the present paper.

In addition to deriving the current flow patterns for a number of cases of interest, particular attention is paid to the Hall potential. For the Hartmann flow problem it was shown earlier[7] that inclusion of the nonequilibrium effect increased the Hall potential by a very small amount. In the present work, on the other hand, one finds that including this effect produces a notable decrease in the Hall potential. A similar reduction has also been predicted by Kerrebrock[8] for a somewhat different theoretical model.

A complete description of the analysis, numerical results, and a discussion of their significance is presented in the following sections.

II. ANALYSIS

The problem to be considered is the uniform flow of a nonequilibrium plasma through the constant-height channel shown in Fig. 1. In reality the duct is of rectangular cross section, but we assume its extent in the z direction to be infinite and restrict our attention to a two-dimensional problem. Each pair of electrodes (one in the lower wall and the one directly above it) is assumed to be connected to a separate resistive load (generator) or power

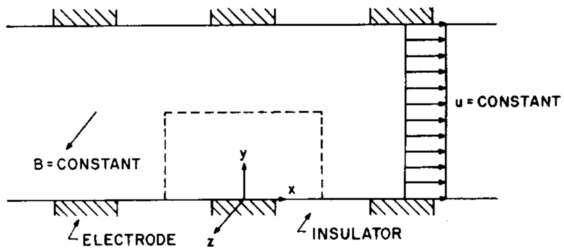

FIG. 1. Constant height magnetohydrodynamic channel with uniform flow and finite electrode segments.

supply (accelerator). In this way no net current can flow axially due to the Hall effect and a Hall potential will be established. The magnetic field is assumed to be constant and equal to that applied, so that we are assuming the magnetic Reynold's number to be negligibly small.

Due to periodicity and symmetry we can solve the problem in the region defined by the dotted line in Fig. 1. This region is shown in more detail in Fig. 2 with a sketch of the current distribution added to illustrate the type of results anticipated. Here we have defined the following quantities: c is the electrode width, l is the electrode pitch, and h is the channel height.

The basic equations to be solved are the "generalized" Ohm's law and Maxwell's equations, since the flow velocity was assumed to be uniform and constant. It must of course be recognized that the Lorentz forces due to the currents flowing will in fact perturb the flow field. For the present we neglect such flow perturbations and assume they can be evaluated by an iterative procedure.[9] We then write the governing relationships for the non-time-varying case.

Generalized Ohm's Law

$$\mathbf{i} = \sigma[\mathbf{E} + \mathbf{u} \times \mathbf{B} - \kappa(\mathbf{i} \times \mathbf{B})]; \tag{1}$$

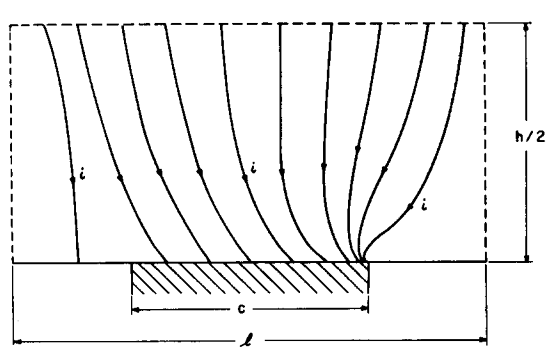

FIG. 2. Current pattern in vicinity of electrode.

[6] A. Sherman, J. Fluid Mech. 25, 621 (1966).
[7] A. Sherman, AIAA J. 3, 783 (1965).
[8] J. L. Kerrebrock, in Proceedings of the Sixth Symposium on Engineering Aspects of Magnetohydrodynamics (1965).

[9] G. W. Sutton and A. W. Carlson, J. Fluid Mech. 11, 121 (1961).

Maxwell's Equations

$$\nabla \times \mathbf{E} = 0, \tag{2}$$

$$\nabla \cdot \mathbf{i} = 0; \tag{3}$$

where $\mathbf{i}$ is the current density, σ the electrical conductivity, $\mathbf{E}$ the electric field intensity, $\mathbf{u}$ the flow velocity, $\mathbf{B}$ the magnetic induction, and $\kappa = 1/en_e$ the Hall coefficient with e the charge on an electron and n_e the electron density. In Eq. (1) currents due to pressure or temperature gradients have been neglected, and the current has been assumed to be solely due to electron flow so that ion slip has been neglected.

For the two-dimensional problem being considered, and u and B constant, we have

$$i_x = \sigma[E_x - \kappa B i_y], \tag{4}$$

$$i_y = \sigma[E_y - uB + \kappa B i_x], \tag{5}$$

$$\partial E_x/\partial y = \partial E_y/\partial x, \tag{6}$$

$$\partial i_x/\partial x + \partial i_y/\partial y = 0. \tag{7}$$

Solving Eqs. (4) and (5) for E_x and E_y and defining a current stream function as

$$i_x = \partial z/\partial y, \qquad i_y = -\partial z/\partial x, \tag{8}$$

we can write

$$E_x = \frac{\partial z/\partial y - \omega\tau(\partial z/\partial x)}{\sigma}, \tag{9}$$

$$E_y = uB - \frac{\partial z/\partial x + \omega\tau(\partial z/\partial y)}{\sigma}, \tag{10}$$

where we have defined $\omega\tau = \sigma\kappa B$. Substituting these in Eq. (6) then yields the equation for z which must be solved. Thus,

$$\frac{\partial^2 Z}{\partial X^2} + \frac{\partial^2 Z}{\partial Y^2} - \frac{\partial Z/\partial Y - \omega\tau(\partial Z/\partial X)}{\tilde{\sigma}} \frac{\partial \tilde{\sigma}}{\partial Y} - \frac{\partial Z/\partial X + \omega\tau(\partial Z/\partial Y)}{\tilde{\sigma}} \frac{\partial \tilde{\sigma}}{\partial X} = 0, \tag{11}$$

where we have assumed $\omega\tau$ uneffected by the nonequilibrium ionization although σ is and have defined the following nondimensional variables:

$$Z = \frac{z}{\sigma_0 uBc}, \qquad X = \frac{x}{c}, \qquad Y = \frac{y}{c}, \qquad \tilde{\sigma} = \frac{\sigma}{\sigma_0}.$$

The procedure for solution will be a perturbation expansion. To introduce the expansion parameter, we define $\sigma(|\mathbf{i}|)$ as

$$\sigma = \sigma_0 + C\,|i| \tag{12}$$

or, in terms of dimensionless quantities,

$$\tilde{\sigma} = 1 + C'[(\partial Z/\partial Y)^2 + (\partial Z/\partial X)^2]^{\frac{1}{2}}. \tag{13}$$

Where we have defined $C' \equiv cuB$ and written the dimensionless current as $i/\sigma_0 uB$. Now C' is taken to be our expansion parameter and is assumed close to zero. We can then expand the solution as follows:

$$Z = Z_0 + C'Z_1 + (C')^2 Z_2 + \cdots .$$

Substituting this into Eq. (11) and equating coefficients of each power of C' to zero, we obtain an infinite set of linear partial differential equations. The zeroth ($[C']^0$)-order and first ($[C']^1$)-order equations are then

$$\partial^2 Z_0/\partial X^2 + \partial^2 Z_0/\partial Y^2 = 0, \tag{14}$$

which is Laplace's equation, and

$$\partial^2 Z_1/\partial X^2 + \partial^2 Z_1/\partial Y^2 = F(Z_0), \tag{15}$$

where

$$F(Z_0) = \left(\frac{\partial Z_0}{\partial Y} - \omega\tau \frac{\partial Z_0}{\partial X}\right) \cdot \frac{(\partial Z_0/\partial Y)(\partial^2 Z_0/\partial Y^2) + (\partial Z_0/\partial X)(\partial^2 Z_0/\partial X\,\partial Y)}{[(\partial Z_0/\partial X)^2 + (\partial Z_0/\partial Y)^2]^{\frac{1}{2}}} + \left(\frac{\partial Z_0}{\partial X} + \omega\tau \frac{\partial Z_0}{\partial Y}\right) \cdot \frac{(\partial Z_0/\partial Y)(\partial^2 Z_0/\partial X\,\partial Y) + (\partial Z_0/\partial X)(\partial^2 Z_0/\partial X^2)}{[(\partial Z_0/\partial X)^2 + (\partial Z_0/\partial Y)^2]^{\frac{1}{2}}},$$

which is Poisson's equation.

To establish the boundary conditions, we assume $E_x \equiv 0$ along an electrode and $i_y \equiv 0$ along an insulator. Then

Electrode

$$\frac{\partial Z_0}{\partial Y} = \omega\tau \frac{\partial Z_0}{\partial X}, \qquad \frac{\partial Z_1}{\partial Y} = \omega\tau \frac{\partial Z_1}{\partial X};$$

Insulator

$$Z_0 = \text{const}, \qquad Z_1 = \text{const};$$

and the constants Z_0 and Z_1 can be different.

To determine appropriate values for Z_0 and Z_1, let us consider the total current passing through an electrode,

$$\begin{aligned} \mathcal{I} &= \int_{-c/2}^{c/2} i_y(y=0)\,dx = -\int_{-c/2}^{c/2} \frac{\partial z}{\partial x}(y=0)\,dx \\ &= -[z(\tfrac{1}{2}c, 0) - z(-\tfrac{1}{2}c, 0)] \\ &= \sigma_0 uBc(Z_L - Z_R) = \sigma_0 uBc\,\Delta Z. \end{aligned}$$

Now, according to our model the extent of nonequilibrium ionization depends on the local current density. Therefore, the average current density along the channel center line should be the same for all callculations in order that valid comparisons may be made. The current density near the electrodes will properly depend on geometry, $\omega\tau$, and C'.

This average current density can be expressed as

$$\langle I\rangle = \mathcal{J}/l = \sigma_0 uBc\,\Delta Z/l.$$

Thus, to maintain the desired reference condition, we keep $(\Delta Z)(l/c)^{-1}$ constant at all times.

III. NUMERICAL SOLUTION

Solutions can now be carried out of Eqs. (14) and (15) for several values of the parameters l/h, l/c, and $\omega\tau$. These will all be valid for any value of C' as long as C' is small compared to unity.

To obtain the required constant value of $\Delta Z/(l/c)$ we select $\Delta Z_1 \equiv 0$ and let ΔZ_0 satisfy the above condition. This choice is essential since $\Delta Z = \Delta Z_0 + C'\Delta Z_1$, and we could not keep ΔZ constant for all C' unless this choice were made.

Also, to compute $\mathcal{E}_x$ and $\mathcal{E}_y$ from the solution obtained in terms of currents, we have

$$\mathcal{E}_x \equiv E_x/uB = (I_x + \omega\tau I_y)/\bar{\sigma}, \tag{16}$$

$$\mathcal{E}_y \equiv E_y/uB = (I_y - \omega\tau I_x)/\bar{\sigma} + 1. \tag{17}$$

Finally, Eqs. (14) and (15) were solved numerically by putting them into finite difference form and iterating for the solution on a high-speed digital computer. The singular points at the downstream edge of the electrodes caused some difficulty. To permit a solution in reasonable time, a rectangular finite difference grid was programmed so that a high density of grid points could be located near the singular point. In this way we were able to reduce the grid size sufficiently so that errors in the solution near the singular point had a negligible effect on the solution throughout the bulk of the region of interest. Typically, 600 grid points were involved in the solution for a particular case. To speed convergence and, thereby, minimize the number of iterations we used the "extrapolated Liebmann" method[10] to obtain an acceleration factor with which to modify the finite difference formula.

IV. RESULTS

Following these procedures solutions have been obtained for different values of the parameters of interest, where for all cases we have kept $\Delta Z/(l/c) = 1$.

[10] S. P. Frankel, Math. Tables Aids Comput. 4, 65 (1950).

In Fig. 3 the current streamlines are shown for one particular case. The solid lines are for $C' = 0$ and the dotted lines represent the $C' = 0.1$ case. In Figs. 4–6 the Hall potential is shown as a function of $\omega\tau$, pitch to channel height ratio, and pitch to electrode width ratio. Figures 7–9 show curves of channel conductance as a function of the same three parameters. The channel conductance is calculated from the solution obtained in the following way.

$$\sigma_{\text{equiv}} = \frac{\langle I\rangle}{h^{-1}\int_0^h (E_y - uB)\,dy} = \frac{(\sigma_0 uB\,\Delta Z)(l/c)^{-1}}{(uBc/h)\int_0^{h/c}(\mathcal{E}_y)\,dY - uB}$$

or

$$\frac{\sigma_{\text{equiv}}}{\sigma_0} = \frac{(\Delta Z)}{(l/h)[\Delta\phi_y - h/c]},$$

where

$$\Delta\phi_y \equiv \int_0^{h/c} \mathcal{E}_y\,dY.$$

V. DISCUSSION

The current streamlines as shown in Fig. 3 illustrate the typical current crowding at one corner of the electrode due to the Hall effect. When nonequilibrium ionization is also taken into account, we see that this crowding is increased. This is reasonable since the plasma resistance is lowest in the region of highest current density, and current should flow preferentially through the low-resistance regions.

The Hall potentials calculated from our solutions are shown in Figs. 4–6 to always be reduced below what they would have been for a constant conductivity calculation. Note that this result is independent of the value of σ_0. Thus, nonequilibrium ionization tends to reduce the Hall potential even when

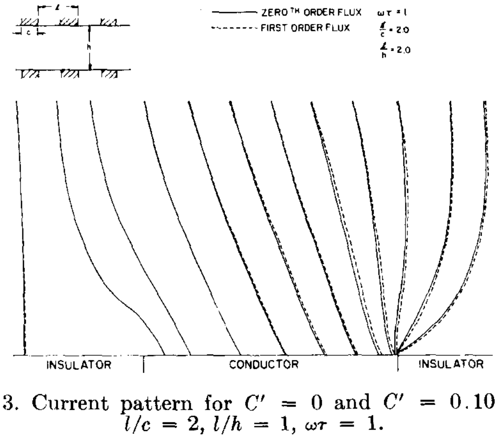

FIG. 3. Current pattern for $C' = 0$ and $C' = 0.10$ when $l/c = 2$, $l/h = 1$, $\omega\tau = 1$.

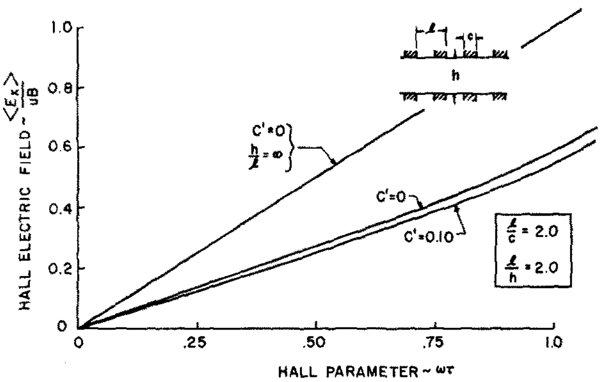

FIG. 4. Average Hall electric field versus Hall parameter for $l/c = 2$, $l/h = 2$.

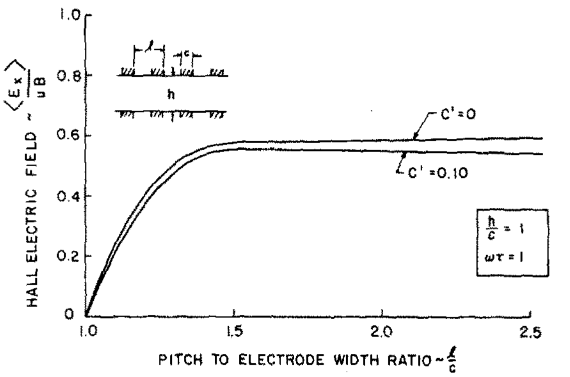

FIG. 6. Average Hall electric field versus pitch to electrode width ratio for $\omega\tau = 1$, $h/c = 1$.

there is no current leakage between electrode pairs. When current leakage does exist, however, there must always be a Hall potential reduction, the only question being of what magnitude. An analysis in which current leakage always exists has in fact been carried out[8] by assuming a uniform high conductivity plasma layer to exist along each wall due to plasma convection from local high-conductivity regions. The main body of plasma is then of uniform lower conductivity. Within this assumption severe Hall potential reductions are predicted. The principal advantage of the uniform conductivity (main body and layers) assumption, from the point of view of the solution of the equations, as can be seen from Eq. (11) is that the equation is linearized. In the present work a Hall potential reduction is predicted without the necessity of choosing an arbitrary layer of high conductivity and, by virtue of this, postulating a current leakage. We might add, however, that our problem if extended to other geometries or values of $\omega\tau$ than were chosen here for convenience would also lead to current leakage.

The line corresponding to a constant conductivity and infinitely fine segmentation is included in Fig. 4 to illustrate the severe Hall potential reduction due to finite electrode segmentation alone. Calculations for $\omega\tau$ values much higher than one were not carried out due to numerical difficulties. From Fig. 6 we recognize that there's little further gain in reducing the electrode size $[c \rightarrow 0, (l/c) \rightarrow \infty]$ beyond $l/c = 1.5$ for the case chosen. There will in fact even be a lowering of the Hall potential when nonequilibrium ionization is allowed for.

In the final series of curves the effective channel conductance is compared to the constant conductivity σ_0. Since the average current flowing per unit channel length, however, is constant, one can interpret these curves in terms of the Faraday potential E_y. In other words, a reduction in σ_{equiv} implies an increase in $(E_y)_{\text{equiv}}$. Thus, as we can see the Faraday potential will always be larger than its ideal value due to finite electrodes. Physically, this situation exists because of the high-resistance paths caused by the current crowding, and the higher potential needed to achieve a specified current flow. We also see that nonequilibrium ionization

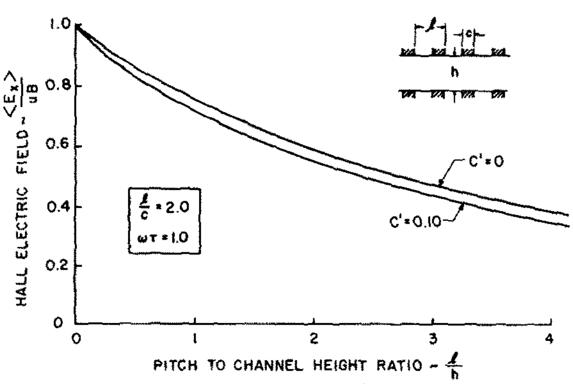

FIG. 5. Average Hall electric field versus pitch to channel height ratio for $\omega\tau = 1$, $l/c = 2$.

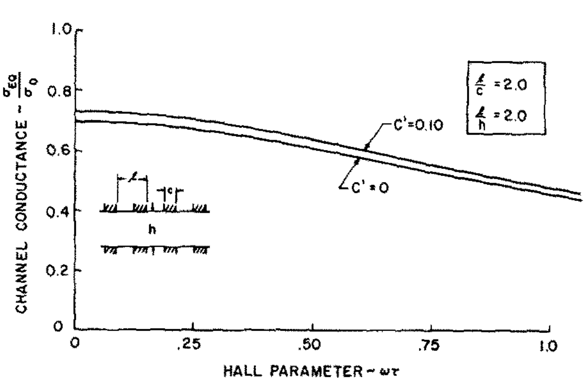

FIG. 7. Channel conductance versus Hall parameter for $l/c = 2$, $l/h = 2$.

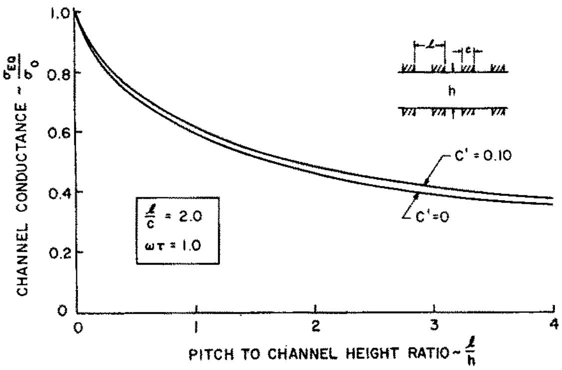

FIG. 8. Channel conductance versus pitch to channel height ratio for $\omega\tau = 1$, $l/c = 2$.

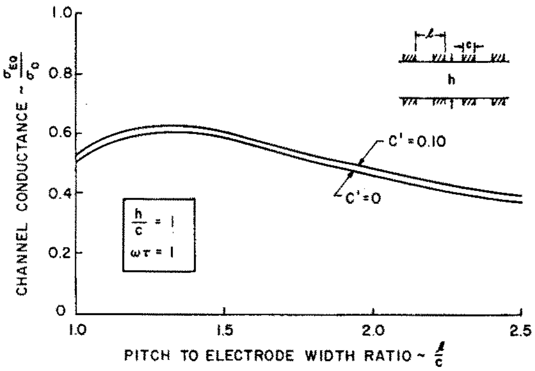

FIG. 9. Channel conductance versus pitch to electrode width ratio for $\omega\tau = 1$, $h/c = 1$.

relieves the situation somewhat. This is misleading, however, since we are comparing the nonequilibrium flow to a constant conductivity flow but at $\sigma = \sigma_0$. Surely, when nonequilibrium ionization exists, σ must be greater than σ_0 everywhere by definition. A more valid comparison could be made if we could make the comparison with a constant conductivity case but use some average $\sigma > \sigma_0$. For the current levels involved in these calculations, we estimate $\sigma \cong 1.03\sigma_0$, so that all the $C' = 0.10$ curves of Figs. 7–9 should be lowered somewhat. The end result is that the nonequilibrium ionization phenomena has little effect on σ_{equiv} or, alternately, the Faraday potential. There is a considerable influence on the Hall potential, however.

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