Lorentz frame in which the center of mass has zero velocity, that is,

(14.60) $$\sum_j \frac{m_j \mathbf{v}_j}{\sqrt{1-\|\mathbf{v}_j\|^2}} = 0\,,$$

where $\mathbf{v}_j$ is the velocity of the $j^{\text{th}}$ particle in this reference frame, we find immediately that

(14.61) $$m = \sum_j \frac{m_j}{\sqrt{1-\|\mathbf{v}_j\|^2}}\,.$$

Thus we see that the total mass of the system is greater than the sum of the constituent masses, and in fact depends on the motion of the system.

□

## Inversions of space and time

Let $U$ be the space of motions of an isolated relativistic dynamical system. Up until now we have assumed that the *restricted* Poincaré group acts on $U$ while preserving the Lagrange form (13.70).

We also know that the *full* Poincaré group has 4 connected components (13.53). It is generated by the restricted Poincaré group $G$ and by the following two operations, namely, *space inversion*†

(14.62) $$I_s\left(\mathcal{R}\begin{pmatrix}\mathbf{r}\\ t\end{pmatrix}\right) = \mathcal{R}\begin{pmatrix}-\mathbf{r}\\ t\end{pmatrix},$$

and *time reversal*

(14.63) $$I_t\left(\mathcal{R}\begin{pmatrix}\mathbf{r}\\ t\end{pmatrix}\right) = \mathcal{R}\begin{pmatrix}\mathbf{r}\\ -t\end{pmatrix}.$$

(14.64) One might ask whether the *full* Poincaré group too is a dynamical group of $U$. It should be noted that general relativity argues in favor of this. Thus let us consider a symplectic manifold $U$ (not necessarily connected) which admits the full Poincaré group $G'$ as a dynamical group.

It is clear that the restricted Poincaré group $G$ is also dynamical group of $U$ and even that it is a dynamical group of *each connected component* of $U$ (theorem (1.51)). We know that the action of $G$ has a moment defined up to an additive constant *on each component* of $U$, and that this constant can be chosen such that

(14.65) $$\psi(\underline{a}_U(x)) = \underline{a}_{G^\bullet}(\psi(x)) \qquad \forall\, a \in G\,,\ \forall\, x \in U\,,$$

†Editors' note: also called *parity reversal*.

where $\psi$ is the map that assigns to each point $x$ of $U$ its moment $\mu$. It is clear that $\mu$ is also a moment of the group $G'$ because the definition of moment (11.7) only involves the Lie algebra of the group.

TECHNICAL DISCUSSION: Let $I$ be an arbitrary element of $G'$ and define

$$F(x) \equiv \psi(\underline{I}_U(x)) - \underline{I}_{\mathcal{G}^*}(\psi(x)).$$

The proof of (11.7) shows that $F(x)$ is constant on each component of $U$.

Now let $a$ be an arbitrary element of $G$. Then formula (14.65) shows that

$$\underline{a}_{\mathcal{G}^*}(F(x)) = \psi(\underline{a \times I}_U(x)) - \underline{a \times I}_{\mathcal{G}^*}(\psi(x)),$$

or, writing $b = I^{-1} \times a \times I$,

$$\underline{a}_{\mathcal{G}^*}(F(x)) = \psi(\underline{I \times b}_U(x)) - \underline{I \times b}_{\mathcal{G}^*}(\psi(x)).$$

Theorem (6.35) shows that $b \in G$. Applying (14.65) again we obtain

$$\underline{a}_{\mathcal{G}^*}(F(x)) = F(\underline{b}_U(x)).$$

Theorem (1.51) shows that $\underline{b}_U(x)$ belongs to the same component of $U$ as $x$. Thus $F(\underline{b}_U(x)) = F(x)$, whence

$$\underline{a}_{\mathcal{G}^*}(F(x)) = F(x) \qquad \forall\, a \in G\,, \forall\, x \in U\,.$$

This expresses the fact that the *coboundary* of the torsor $F(x)$ of the group $G$ is *zero* (see definition (11.19)). From this it is elementary to deduce (using, for example, expression (13.107) of the coadjoint representation) that $F(x)$ is zero. □

We thus have proved the identity

(14.66) $$\psi(\underline{I}_U(x)) = \underline{I}_{\mathcal{G}^*}(\psi(x)) \qquad \forall\, x \in U\,,\ \forall\, I \in G'\,,$$

which extends the validity of (14.65) to the full Poincaré group.

This formula applies when $I = I_s$ (space inversion) or when $I = I_t$ (time reversal). Calculating $\underline{I_s}_{\mathcal{G}^*}$ and $\underline{I_t}_{\mathcal{G}^*}$ gives (notation (13.59)):

(14.67) $$\begin{cases} \underline{I_s} : \mathbf{l} \to \mathbf{l}\,, & \mathbf{g} \to -\mathbf{g}\,, \quad \mathbf{p} \to -\mathbf{p}, \quad E \to E\,, \\ \underline{I_t} : \mathbf{l} \to \mathbf{l}\,, & \mathbf{g} \to -\mathbf{g}\,, \quad \mathbf{p} \to \mathbf{p}\,, \quad E \to -E\,. \end{cases}$$

(14.68) Let us now extend definition (14.1) of an elementary system by postulating that it is the full Poincaré group $G'$ which acts transitively and canonically on $U$. □

(14.69) From (14.66) it follows immediately that the moment $\mu \equiv \psi(x)$ varies over a coadjoint orbit *of* $G'$ when $x$ varies over $U$. If this orbit is not connected, then *neither is* $U$ since $\psi$ is continuous (theorem (6.51)).

On the other hand, we know that every orbit in $U$ of the restricted Poincaré group $G$ is contained in a connected component $U_0$ of $U$. Since $G$ and $G'$ have the same Lie algebra, these orbits have the same dimension as $U$ (theorem (6.20)). They thus are pairwise disjoint open sets filling out $U_0$. But this is impossible if there is more than one orbit because $U_0$ is connected. Hence each connected component of $U$ is an orbit of the restricted Poincaré group and thus defines *an elementary system in the restricted sense* (14.1).

(14.70) It follows that the space of motions of an elementary system for the full Poincaré group is obtained by taking the *sum*[263] of the spaces of motions of several elementary systems in the restricted sense (14.1). □

Let us treat some examples.

## A particle with nonzero mass $m$

(14.71) Let us begin by observing that the two Casimirs $\overline{P} \cdot P$ and $\overline{W} \cdot W$ are constant on every coadjoint orbit of $G'$ and hence on $U$ (14.69). If we assume that $\overline{P} \cdot P > 0$, then each connected component of $U$ corresponds to a *particle* of mass $m = \pm\sqrt{\overline{P} \cdot P}$ and spin $s = \sqrt{\frac{-\overline{W} \cdot W}{\overline{P} \cdot P}}$. Equation (14.67) shows that time reversal $I_t$ changes the sign of the energy and thus the sign of the mass (14.4◇). Consequently *it transforms every motion of a particle of mass $m$ into a motion of a particle of mass $-m$.*

On the other hand, space inversion $I_s$ preserves the mass $m$ and the spin $s$. Hence the associated orbit of $G'$ has *two components.*

(14.72) If $\psi$ is bijective, then $U$ has two components as well. But it is possible that $U$ has four components. A model of this possibility can be constructed as follows:

$U$ = the set of pairs $x \equiv (\mu, \varepsilon)$ [$\mu$ lies in a coadjoint orbit and $\varepsilon = \pm 1$]

(14.73) $$\sigma(\delta x)(\delta' x) \equiv \sigma(\delta\mu)(\delta'\mu)$$

$\underline{a}_U(\mu, \varepsilon) = \left(\underline{a}_U(\mu), \chi_s(a)\varepsilon\right)$, where $\chi_s(a)$ is the *spatial character* of $a$.[264] □

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[263]By the *sum* of several sets one means their *union*, provided that their pairwise intersections are empty.

There is a way to safeguard the connectedness of $U$ and thus to avoid particles of negative mass. First one postulates that $\psi$ is injective and then one lets the *full* Poincaré group $G'$ act on the *unique* component where $m > 0$ by defining

(14.74) $$\underline{a}(\mu) = \chi_t(a)\, \underline{a}_{\mathcal{G}^*}(\mu)\,,$$

where $\chi_t(a)$ is the *temporal character* of $a$.[265]

(14.75) However, we know that these transformations cannot be canonical (14.71). More precisely, if $\chi_t(a) = -1$ (one says that $a$ is *antichronous*), the transformation $\underline{a}$ defined by (14.74) is anticanonical (definition (10.18)).

We thus have to modify the axioms of symplectic mechanics by allowing anticanonical transformations to belong to a dynamical group.

(14.76) There is another way to arrive at the same result. It consists of *excluding* the antichronous transformations by looking only at the subgroup $G''$ defined by $\chi_t(a) = +1$. $G''$, which is called the *orthochronous* group, is indeed a dynamical group (without anticanonical elements) of a connected manifold.

It should be obvious that these various conventions have to be judged according to their ability to explain experiments.[266] □

### A massless particle

(14.77) Let us assume that $\overline{P} \cdot P = 0$ and $\overline{W} \cdot W = 0$ while $P$ and $W$ are nonzero. Then each component of $U$ corresponds to a massless particle in the sense of (14.29). These particles have the *same spin* $s$, but the formulæ (14.67) show that space inversion changes the *helicity* and time reversal the *sign of the energy*. It follows that the coadjoint orbit has four components and consequently $U$ as well. Hence $\psi$ is necessarily a bijection.

[264]By definition $\chi_s(a)$ equals $+1$ on the components of the group $G'$ containing 1 and $I_t$ and equals $-1$ on the components containing $I_s$ and $I_t \cdot I_s$. It is called a *character* because it satisfies $\chi_s(ab) \equiv \chi_s(a)\chi_s(b)$.

[265]$\chi_t(a)$ equals 1 on the components of $G'$ containing 1 and $I_s$ and equals $-1$ on the components containing $I_t$ and $I_t \cdot I_s$. It also satisfies $\chi_t(ab) \equiv \chi_t(a)\chi_t(b)$.

[266]As it happens, recent experiments seem to show that the artifice (14.74) does not apply universally in physics. If — as general relativity suggests — it is the ***full*** Poincaré group, that is the dynamical group of real systems, it is not possible to call particles with negative mass into question. Thus one might hope to find them in nature, although statistical mechanics shows us they must be rare (see (17.149)).